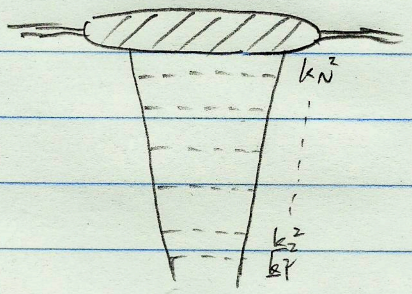
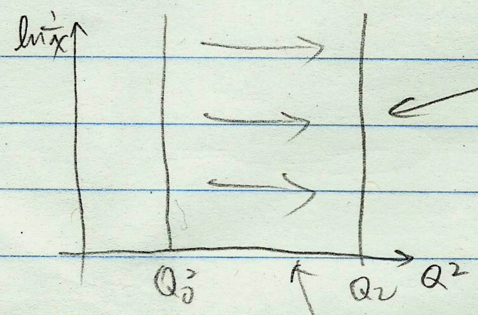
We get a ladder diagram



$$Q^2 \gg \underline{k}_1^2 \gg \underline{k}_2^2 \gg \cdots \gg \underline{k}_N^2 \gg \Lambda_{QCD}^2$$

How does DGLAP work?



③ obtain distribution at Q^2 we want

① ~~start~~ with initial condition ~ function $p^f(x, Q_0^2)$

② and solve the DGLAP equation ("evolve" the distribution function)

DGLAP at small x

At small x, Gluons dominate ~ (neglect quarks).

$$Q^2 \frac{\partial}{\partial Q^2} G(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx'}{x'} \gamma_{GG}\left(\frac{x}{x'}\right) G(x', Q^2)$$

Consider moments of $x\,G(x, Q^2)$:

$$G_n(Q^2) = \int_0^1 dx\, x^{n-1}\, G(x, Q^2)$$

such that $G(x, Q^2) = \int \frac{dn}{2\pi i} x^{-n} G_n(Q^2)$

$$Q^2 \frac{\partial}{\partial Q^2} G_n(Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_0^1 dx\, x^{n-1} \int_x^1 \frac{dx'}{x'} \gamma_{GG}\left(\frac{x}{x'}\right) G(x', Q^2)$$

$$= \frac{\alpha(Q^2)}{2\pi} \int_0^1 dx' (x')^{n-1} G(x', Q^2) \int_0^{x'} \frac{dx}{x'} \left(\frac{x}{x'}\right)^{n-1} \gamma_{GG}\left(\frac{x}{x'}\right)$$

$z = \frac{x}{x'}$

$$= \frac{\alpha(Q^2)}{2\pi} \underbrace{\int_0^1 dx' (x')^{n-1} G(x', Q^2)}_{G_n(Q^2)} \underbrace{\int_0^1 dz\, z^{n-1} \gamma_{GG}(z)}_{\gamma_{GG}^{(n)}}$$

$$Q^2\frac{\partial}{\partial Q^2}G_n(Q^2)=\frac{\alpha(Q^2)}{2\pi}\gamma_{GG}^{(n)}G_n(Q^2)$$ DGLAP in Mellin space.

$$\therefore G_n(Q^2)=e^{\int_{Q_0^2}^{Q^2}\frac{dQ'^2}{Q'^2}\frac{\alpha(Q'^2)}{2\pi}\gamma_{GG}^{(n)}}G_n(Q_0^2)$$

$$\alpha(Q^2)=\frac{1}{\beta_2\ln Q^2/\Lambda^2}\qquad \beta_2=\frac{11N_c-2N_f}{12\pi}$$

$$\therefore\int_{Q_0^2}^{Q^2}\frac{dQ'^2}{Q'^2}\frac{\alpha(Q'^2)}{2\pi}=\frac{1}{2\pi\beta_2}\int_{Q_0^2}^{Q^2}\frac{dQ'^2}{Q'^2}\frac{1}{\ln Q'^2/\Lambda^2}$$

$$=\frac{1}{2\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)$$

$$\therefore G_n(Q^2)=\exp\left[\frac{\gamma_{GG}^{(n)}}{2\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]G_n(Q_0^2).$$

$$\Rightarrow G(x,Q^2)=\int\frac{dn}{2\pi i}x^{-n}e^{\frac{\gamma_{GG}^{(n)}}{2\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)}G_n(Q_0^2)$$

at small x, $\gamma_{GG}(z)\approx\frac{2N_c}{z}$ (small z)

$$\gamma_{GG}^{(n)}\approx\int_0^1 dz\, z^{n-2}\cdot 2N_c=\frac{2N_c}{n-1}\qquad(\text{for } n>1)$$

Now, we evaluate $G(x,Q^2)$ in the saddle point approximation

$$G(x,Q^2)=\int\frac{dn}{2\pi i}\exp\left[n\ln\frac{1}{x}+\frac{N_c}{n-1}\frac{1}{\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]G_n(Q_0^2)$$

The saddle point

$$\frac{d}{dn}\left[n\ln\frac{1}{x}+\frac{N_c}{n-1}\frac{1}{\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]\Bigg|_{n=n_0}=0$$

$$\Rightarrow n_0-1=\pm\sqrt{\frac{N_c}{\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\frac{1}{\ln\frac{1}{x}}}$$ ("+" sign gives dominant contribution to $(n_0-1)\ln\frac{1}{x}$)

$$\therefore xG(x,Q^2)=G_{n_0}(Q_0^2)\cdot e^{2\sqrt{\frac{N_c}{\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\ln\frac{1}{x}}}\frac{1}{\sqrt{4\pi}}\left(\frac{N_c}{\pi\beta_2}\right)^{1/4}\ln^{-3/4}\frac{1}{x}$$
$$\times\left[\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]^{1/4}$$

Mellin Transformation.

The moment $f_\omega(Q^2)$ of a distribution function $f(x, Q^2)$ is defined by the Mellin transform.

$$f_\omega(Q^2) = \int_0^1 dx\, x^\omega f(x,Q^2)$$

and.

$$f(x,Q^2) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i}\, x^{-\omega-1} f_\omega(Q^2)$$

Where the integral in ω-space runs along a contour parallel to the imaginary axis and to the right of all the singularities of the moment $f_\omega(Q^2)$ (which is adjusted by the arbitrary real number a).

check:

$$\int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i}\, x^{-\omega-1} \cdot \int_0^1 d\tilde{x}\, \tilde{x}^\omega f(\tilde{x}, Q^2).$$

$$= \int_0^1 d\tilde{x}\, f(\tilde{x}, Q^2) \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \frac{\tilde{x}^\omega}{x^{\omega+1}} = \int_0^1 d\tilde{x}\, f(\tilde{x}, Q^2) \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left(\frac{\tilde{x}}{x}\right)^\omega \frac{1}{x}$$

$$= \int_0^1 d\tilde{x}\, f(\tilde{x}, Q^2) \frac{1}{x} \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i}\, e^{\omega \ln(x'/x)}$$

let $\omega = a + i\lambda$

$$= \int_0^1 d\tilde{x}\, f(\tilde{x}, Q^2) \frac{1}{x} e^{a \ln(\tilde{x}/x)} \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi}\, e^{i\lambda \ln(\tilde{x}/x)}$$

$$= \int_0^1 d\tilde{x}\, f(\tilde{x}, Q^2) \frac{1}{x} e^{a \ln(\tilde{x}/x)}\, \delta(\ln(\tilde{x}/x))$$

$$= \int_0^1 d\tilde{x}\, f(\tilde{x}, Q^2) \frac{1}{x} e^{a \ln(\tilde{x}/x)}\, \tilde{x}\,\delta(\tilde{x} - x).$$

$$= f(x, Q^2). \quad (\mathrm{V})$$

$$Q^2 \frac{\partial}{\partial Q^2} G(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx'}{x'} \gamma_{GG}\left(\frac{x}{x'}\right) G(x', Q^2)$$

$$\therefore Q^2 \frac{\partial}{\partial Q^2} G_\omega(Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_0^1 dx\, x^\omega \int_x^1 \frac{dx'}{x'} \gamma_{GG}\left(\frac{x}{x'}\right) G(x', Q^2) \qquad (x \leq x')$$

$$= \frac{\alpha(Q^2)}{2\pi} \int_0^1 dx' (x')^\omega G(x', Q^2) \int_0^{x'} \frac{dx}{x'} \left(\frac{x}{x'}\right)^\omega \gamma_{GG}\left(\frac{x}{x'}\right)$$

let $z = \frac{x}{x'}$,

$$= \frac{\alpha(Q^2)}{2\pi} \int_0^1 dx' (x')^\omega G(x', Q^2) \int_0^1 dz\, z^\omega \gamma_{GG}(z)$$

$$= \frac{\alpha(Q^2)}{2\pi} \cdot G_\omega(Q^2) \cdot \gamma_{GG}(\omega)$$

$$\therefore Q^2 \frac{\partial}{\partial Q^2} G_\omega(Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \gamma_{GG}(\omega)\, G_\omega(Q^2).$$

$$\therefore G_\omega(Q^2) = \exp\left[\int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{\alpha(Q'^2)}{2\pi} \gamma_{GG}(\omega)\right]$$

At small x, the z-integral in the splitting functions may get extra enhancement from the small-z region.

At small z, only two splitting functions have singularities,

$$P_{Gq}(z)\Big|_{z \ll 1} \approx \frac{2C_F}{z}, \quad P_{GG}(z)\Big|_{z \ll 1} \approx \frac{2N_c}{z}$$

So we conclude that the evolution of the gluon distribution $G(x, Q^2)$ runs much faster than that of the quark distributions at small x.

So by neglecting quark distribution,

$$Q^2 \frac{\partial G(x, Q^2)}{\partial Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \frac{2N_c}{z} G\left(\frac{x}{z}, Q^2\right)$$

Before solving the equation, let us clarify the approximation that we have made,

$$Q^2 \frac{\partial x G(x, Q^2)}{\partial Q^2} = \frac{\alpha_s(Q^2) N_c}{\pi} \int_x^1 \frac{dx'}{x'} x' G(x', Q^2).$$

$$\therefore \frac{\partial^2 x G(x, Q^2)}{\partial \ln\left(\frac{1}{x}\right) \partial \ln(Q^2/Q_0^2)} = \frac{\alpha_s(Q^2) N_c}{\pi} x G(x, Q^2)$$

For a fixed coupling constant, the solution iterates power of α_s with $\ln\frac{1}{x}\ln\frac{Q^2}{Q_0^2}$

$\therefore$ the resummation parameter is

$$\alpha_s \ln\frac{1}{x} \ln\frac{Q^2}{Q_0^2}$$

at small coupling $\alpha_s \ll 1$, large $Q^2 \gg Q_0^2$ and small x $\ln(\frac{1}{x}) \gg 1$, this is called double logarithmic approximation (DLA)

Now, let's solve the equation,

$$\gamma_{GG}(\omega) = \int_0^1 dz\, z^{\omega} \frac{2N_c}{z} = \frac{2N_c}{\omega} z^{\omega}\Big|_0^1 \approx \frac{2N_c}{\omega} \qquad (\omega > 0)$$

$$\therefore Q^2 \frac{\partial G_\omega(Q^2)}{\partial Q^2} = \frac{\alpha_s(Q^2) N_c}{\pi} \frac{1}{\omega} G_\omega(Q^2)$$

$$\therefore G_\omega(Q^2) = \exp\left\{ \int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{\alpha_s(Q'^2) N_c}{\pi\omega} \right\} G_\omega(Q_0^2)$$

$$\therefore xG(x,Q^2) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} x \frac{1}{x^{\omega+1}} G_\omega(Q^2)$$

$$= \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \exp\left\{ \omega \ln\frac{1}{x} + \int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{\alpha_s(Q'^2) N_c}{\pi\omega} \right\} G_\omega(Q_0^2)$$

With one-loop running coupling constant

$$\alpha_s(Q^2) = \frac{1}{\beta_2 \ln(Q^2/\Lambda^2)}$$

With $Q_0^2 \gg \Lambda^2$,

$$xG(x,Q^2) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \exp\left\{ \omega \ln\frac{1}{x} + \frac{N_c}{\pi\beta_2\omega} \ln\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)} \right\} G_\omega(Q_0^2)$$

$$\int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{\alpha(Q'^2)}{2\pi} = \frac{1}{2\pi\beta_2} \int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{1}{\ln Q'^2/\Lambda^2} = \frac{1}{2\pi\beta_2} \int_{Q_0^2}^{Q^2} d(\ln Q'^2) \frac{1}{\ln Q'^2/\Lambda^2}$$

$$= \frac{1}{2\pi\beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2} \right).$$

Without know the explicit form of $G_\omega(Q_0^2)$, $xG(x,Q^2)$ can be evaluated through saddel point approximation at very small x and large Q^2

To do so, we rewrite equation as

$$xG(x,Q^2) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} e^{P(\omega)} G_\omega(Q_0^2)$$

$$P(\omega) = \omega \ln\frac{1}{x} + \frac{N_c}{\pi\beta_2\,\omega} P(Q^2) \qquad P(Q^2) = \ln \frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}$$

$$= \ln \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}$$

At small x, $\ln\frac{1}{x} \gg 1$, so the integrand oscillate crazy with ω. The oscillation are not there only at the saddle point

$$\left.\frac{\partial}{\partial\omega} P(\omega)\right|_{\omega_{sp}} = 0 \Rightarrow \omega_{sp} = \pm\sqrt{\frac{N_c}{\pi\beta_2}\frac{P(Q^2)}{\ln(\frac{1}{x})}}$$

The "+" sign dominate the integrand (for $\omega_{sp}<0$, $e^{\omega_{sp}\ln\frac{1}{x}}$ strongly suppress the integrand)

$$\therefore P(\omega) \approx P(\omega_{sp}) + \frac{1}{2} P''(\omega_{sp})(\omega-\omega_{sp})^2$$

$$\int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} e^{\left(P(\omega_{sp}) + \frac{1}{2}P''(\omega_{sp})(\omega-\omega_{sp})^2\right)} G_\omega(Q_0).$$

take $i\delta = \omega - \omega_{sp}$, and $a = \omega_{sp}$.

$$\int_{-\infty}^{+\infty} \frac{d\delta}{2\pi} e^{P(\omega_{sp}) - \frac{1}{2}P''(\omega_{sp})\delta^2} G_{\omega_{sp}}(Q_0)$$

$$= \frac{G_{\omega_{sp}}(Q_0^2)}{\sqrt{2\pi P''(\omega_{sp})}} e^{P(\omega_{sp})} = xG(x,Q^2).$$

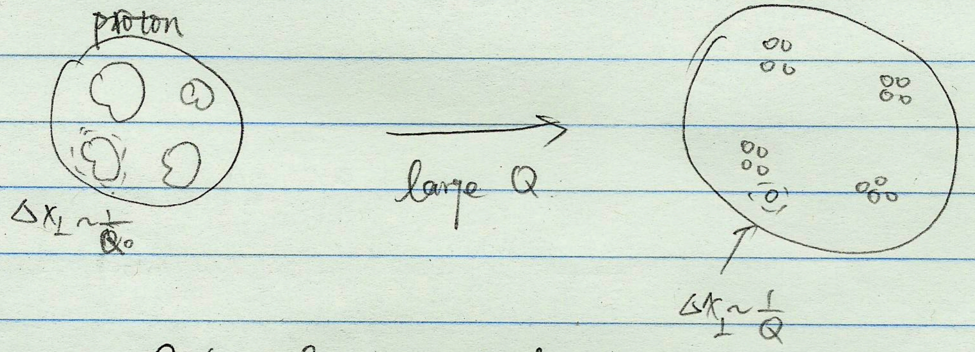
$$\therefore xG(x,Q^2) = \frac{G_{\omega_{sp}}(Q_0^2)}{\sqrt{4\pi}} \left\{\frac{N_c}{\pi\beta_2} \ln\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right\}^{1/4} \left(\ln\frac{1}{x}\right)^{-3/4}$$

$$\times \exp\left\{2\sqrt{\frac{N_c}{\pi\beta_2}\ln\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\ln\frac{1}{x}}\right\}.$$

$$\therefore xG(x,Q^2) \propto \exp\left\{2\sqrt{\frac{N_c}{\pi\beta_2}\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\ln\frac{1}{x}}\right\}.$$

Therefore, $\quad xG \sim e^{2\sqrt{\frac{N_c}{\pi\beta_2}\ln\frac{1}{x}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)}}$.

xG grows at small x, slower than a power of x but faster than any power of $\ln\frac{1}{x}$.



We increase Q (resolution) and see more partons.

BFKL evolution

*1. Paradigm shift

We are going to study high energy behavior of QCD. The small-x asymptotics is synonymous with the high energy limit of QCD (DIS, $x = \frac{Q^2}{s+Q^2}$).

$$\text{small } x \Longleftrightarrow \text{high energy } s.$$

The small-x asymptotics of gluon distribution function at fixed coupling α_s.

$$xG(x,Q^2) \propto \exp\left(2\sqrt{\frac{\alpha_s N_c}{\pi}\ln\frac{1}{x}\ln\frac{Q^2}{Q_0^2}}\right).$$

and corresponding resummation of the parameter is

$$\alpha_s \ln\frac{1}{x}\ln\frac{Q^2}{Q_0^2}$$

The resulting gluon distribution grows with decreasing x in such a way that,

$$\left(\frac{1}{x}\right)^a \gg xG(x,Q^2) \propto \exp\left(2\sqrt{\frac{\alpha_s N_c}{\pi}\ln\frac{1}{x}\ln\frac{Q^2}{Q_0^2}}\right) \gg \ln^n\frac{1}{x}.$$

Which is faster than any positive power n of $\ln\frac{1}{x}$, but slower than any power a of $\frac{1}{x}$. This behavior is valid in the double logarithmic limit of small x and large Q^2. However, if one is interested in studying the high energy limit of QCD at Q^2 of some not necessarily large value and small-x asymptotics. The $\ln Q^2/Q_0^2 \sim 1$, and aimed to resum

$$\alpha_s \ln\frac{1}{x} \sim 1$$

Resummation of a series in powers of the parameter $\alpha_s \ln\frac{1}{x}$ is referred to as the leading-logarithmic approximation in $\ln\frac{1}{x}$.

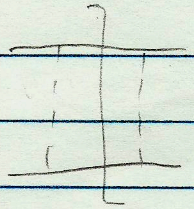
In x-evolution, we hope to find the number of partons of roughly the same transverse size at low x if we know this number at some x_0.

Two-gluon exchange: the low-Nussinov pomeron

Consider high energy scattering event in which a particle of spin j is exchanged in the t-channel between some scatters, the rule is: If one wants to count the powers of the center-of-mass energy s in the total scattering cross section, then the contribution of each t-channel exchange of particle with spin j to the scattering cross section is

$$s^{j-1}$$

For exchange of two t-channel particle of spin j.



$$\sigma \sim s^{2(j-1)}$$

For gluon $j=1$, $\sigma_{gluon} \sim s^0$.

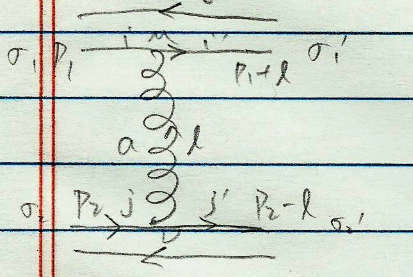
We would expect the cross section due to two-gluon exchange to be constant with energy.

For quark, $j=\frac{1}{2}$.

$$\sigma_{quark} \sim \frac{1}{s} \quad \text{(decrease with } s\text{)}$$

So gluon contribution to the scattering cross section dominates the quark contribution.

At high energy, the dominant lowest-order contribution to the QCD scattering amplitude is due to a t-channel gluon exchange.



$$i\mathcal{M}^0_{qq-qq} = \bar{u}_{\sigma_1'}(p_1+l)\,(ig\gamma^\mu(t^a)_{i'i})\,u_{\sigma_1}(p_1)$$
$$\times \frac{-i}{l^2}\, g_{\mu\nu}$$
$$\times \bar{u}_{\sigma_2'}(p_2-l)\,(ig\gamma^\nu(t^a)_{j'j})\,u_{\sigma_2}(p_2)$$
$$= +ig^2(t^a)_{i'i}(t^a)_{j'j}\,\frac{1}{l^2}\,\bar{u}_{\sigma_1'}(p_1+l)\gamma^\mu u_{\sigma_1}(p_1)$$
$$\times \bar{u}_{\sigma_2'}(p_2-l)\gamma_\mu u_{\sigma_2}(p_2)$$

From the onshell condition,

$$(p_1+l)^2 = 2p_1\cdot l \approx p_1^+ l^- + p_1^- l^+ - 2\vec{p}_{1\perp}\cdot\vec{l}_\perp = 0 \Rightarrow l^- = O\left(\frac{1}{p_1^+}\right)$$

$$(p_2-l)^2 = -2p_2\cdot l = -p_2^- l^+ - p_2^+ l^- + 2\vec{p}_{2\perp}\cdot\vec{l}_\perp = 0 \Rightarrow l^+ = O\left(\frac{1}{p_2^-}\right)$$

$$\therefore\ l^2 = l^+ l^- - l_\perp^2 \approx -l_\perp^2$$

$$\therefore\ iM^0_{qq\to qq} = +i g^2 (t^a)_{i'i}(t^a)_{j'j}\frac{1}{-l_\perp^2}\cdot 2p_1^\mu\cdot 2p_{2\mu}\,\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}$$

$$= 4i g^2 (t^a)_{i'i}(t^a)_{j'j}\frac{1}{l_\perp^2}\frac{1}{2}p_1^+ p_2^-\,\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}$$

$$\downarrow s = (p_1+p_2)^2 = 2p_1\cdot p_2 = p_1^+ p_2^-$$

$$M^0_{qq\to qq} = -2g^2(t^a)_{i'i}(t^a)_{j'j}\,\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}\frac{s}{l_\perp^2}$$

$$\sigma^0_{qq\to qq} = \frac{1}{2E_1 2E_2|v_1-v_2|}\frac{1}{4}\frac{1}{N_c^2}\sum_{\substack{\sigma_1,\sigma_2\\ \sigma_1',\sigma_2'}}\int\frac{d^3p_1'}{(2\pi)^3 2E_1'}\frac{d^3p_2'}{(2\pi)^3 2E_2'}(2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2')$$

$$\times |M(p_1,p_2\to p_1',p_2')|^2$$

$$\frac{1}{4}\sum_{\substack{\sigma_1,\sigma_2\\ \sigma_1',\sigma_2'}}|M|^2 = 4g^4\,\mathrm{Tr}(t^a t^b)\,\mathrm{Tr}(t^a t^b)\frac{s^2}{(l_\perp^2)^2}\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}$$

$$\downarrow \mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$$

$$= 4g^4\frac{1}{4}\cdot\delta^{ab}\delta^{ab}\frac{s^2}{(l_\perp^2)^2} = 4g^4\cdot\frac{1}{4}(N_c^2-1)\frac{s^2}{(l_\perp^2)^2}$$

$$= 4g^4\frac{1}{2}C_F\cdot N_c\frac{s^2}{(l_\perp^2)^2}$$

$$\downarrow \alpha_s = g^2/4\pi$$

$$= 32\pi^2\alpha_s^2\,C_F\cdot N_c\frac{s^2}{(l_\perp^2)^2}$$

$$\therefore\ \sigma^0_{qq\to qq} = \frac{1}{4s\cdot 2}\frac{1}{N_c^2}\int\frac{d^3p_1'}{(2\pi)^3\sqrt{s}}\frac{d^3p_2'}{(2\pi)^3\sqrt{s}}(2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2')$$

$$\times\ 32\pi^2\alpha_s^2 C_F N_c\frac{s^2}{(l_\perp^2)^2}$$

$$\downarrow \frac{d^3p_2'}{2E_2(2\pi)^3} = \frac{d^4p_2'}{(2\pi)^4}(2\pi)\delta((p_2'^0)^2-|\vec{p}_2'|^2)$$

$$= \frac{1}{2s}\frac{1}{N_c^2}32\pi^2\alpha_s^2 C_F N_c\int\frac{d^3p_1'}{(2\pi)^3\sqrt{s}}(2\pi)\delta\left((p_1^0+p_2^0-p_1'^0)^2-|\vec{p}_1+\vec{p}_2-\vec{p}_1'|^2\right)$$

$$\times\ s^2/(l_\perp^2)^2$$

let $p_1' = p_1 + l$.

$$\therefore\ \sigma^{0}_{qq\to qq} = \frac{16\pi^2\alpha_s^2 C_F}{N_c}\int \frac{d^3 l}{(2\pi)^3\,\sqrt{s}}\,(2\pi)\,\delta\left((p_2^0+l^0)^2 - |\vec p_2+\vec l|^2\right)$$

$$\times \frac{s}{(l_\perp^2)^2}$$

$$= \frac{4\alpha_s^2 C_F}{N_c}\,\frac{s}{s}\int \frac{d^2 l_\perp}{(l_\perp^2)^2}$$

? Yuri has $\frac{2\alpha_s^2 C_F}{N_c}$

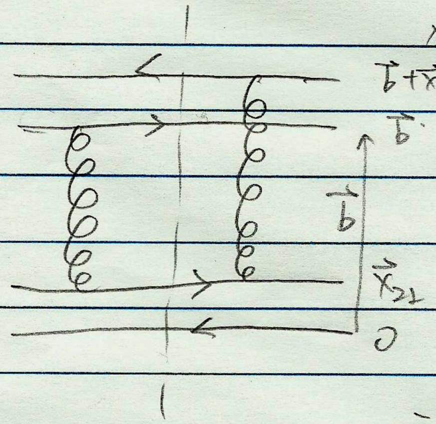
$$= \frac{4\alpha_s^2 C_F}{N_c}\int \frac{d^2 l_\perp}{(l_\perp^2)^2}$$

Which is independent of energy at high energy. The two t-channel gluon exchange cross section is called Low-Nussinov pomeron.

The $l_\perp$-integral has an infrared divergence. This is because we are calculating a cross section for scattering of free color charges. To make the cross section IR-finite, we need to remember that scattering quarks are part of the onium wave functions.

$$\sigma_{tot}^{onium+onium} = \int d^2x_{1\perp}\, d^2x_{2\perp} \int_0^1 dz_1\, dz_2\, |\psi(\vec x_{1\perp}, z_1)|^2\, |\psi(\vec x_{2\perp}, z_2)|^2$$

$$\times\, \hat\sigma_{tot}^{o+o}$$



In $\hat\sigma_{tot}^{onium+onium}$, since we fixed the coordinate of the four quark lines we have factors in coordinate space

$$\int d^2 b\, \left|\left(e^{i\vec l_\perp\cdot\vec b} - e^{i\vec l_\perp\cdot(\vec b+\vec x_1)}\right)\left(1-e^{i\vec l_\perp\cdot\vec x_2}\right)\right|^2$$

$$= \underbrace{\int d^2 b\; e^{i(\vec l_\perp-\vec l_\perp')\cdot\vec b}}_{\delta^{(2)}(\vec l_\perp-\vec l_\perp')}\left(1-e^{i\vec l_\perp\cdot\vec x_1}\right)\left(1-e^{i\vec l_\perp\cdot\vec x_2}\right)$$

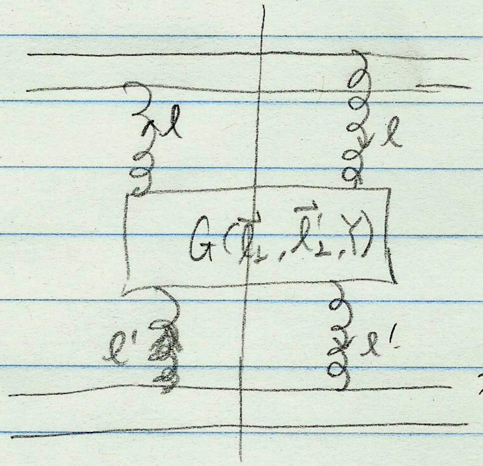
$$\times\left(1-e^{-i\vec l_\perp'\cdot\vec x_1}\right)\left(1-e^{-i\vec l_\perp'\cdot\vec x_2}\right)$$

$$= \left(2-e^{-i\vec l_\perp\cdot\vec x_{1\perp}} - e^{i\vec l_\perp\cdot\vec x_{1\perp}}\right)\left(2-e^{-i\vec l_\perp\cdot\vec x_{2\perp}} - e^{i\vec l_\perp\cdot\vec x_{2\perp}}\right)$$

c. $$\hat{\sigma}_{tot}^{onium+onium} = \frac{2\alpha_s^2 C_F}{N_c}\int \frac{d^2 l_\perp}{(l_\perp^2)^2}\left(2 - e^{-i\vec{l}_\perp\cdot\vec{x}_{1\perp}} - e^{i\vec{l}_\perp\cdot\vec{x}_{1\perp}}\right) \times\left(2 - e^{-i\vec{l}_\perp\cdot\vec{x}_{2\perp}} - e^{i\vec{l}_\perp\cdot\vec{x}_{2\perp}}\right)$$

At IR range, $l_\perp$ is small, we can expand the exponential factors and the leading non-vanish terms comes with $(l_\perp^2)$, which will cancels the denominator and get rid of the IR divergence.

The Balitsky - Fadin - Kuraev - Lipatov evolution equation.



$$\hat{\sigma}_{tot}^{onium+onium} = \frac{2\alpha_s^2 C_F}{N_c}\int \frac{d^2 l_\perp\, d^2 l'_\perp}{l_\perp^2\, l'^2_\perp}$$

$$\times\left(2 - e^{-i\vec{l}_\perp\cdot\vec{x}_{1\perp}} - e^{i\vec{l}_\perp\cdot\vec{x}_{1\perp}}\right)$$

$$\times\left(2 - e^{-i\vec{l}'_\perp\cdot\vec{x}_{2\perp}} - e^{i\vec{l}'_\perp\cdot\vec{x}_{2\perp}}\right)\cdot G(\vec{l}_\perp, \vec{l}'_\perp, Y)$$

We define the rapidity variable $Y = \ln(s\,|\vec{x}_{1\perp}|\,|\vec{x}_{2\perp}|)$

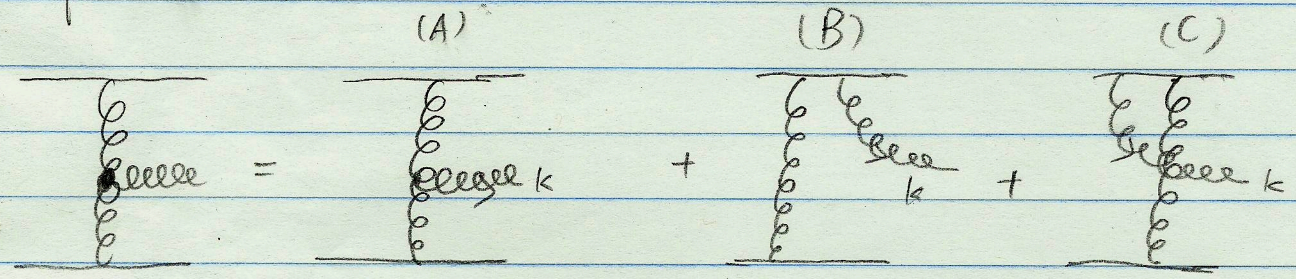
For the lowest order,

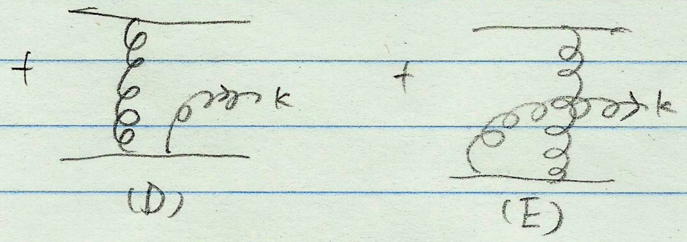
$$G(\vec{l}_\perp, \vec{l}'_\perp, Y=0) = G_0(\vec{l}_\perp, \vec{l}'_\perp) = \delta^{(2)}(\vec{l}_\perp - \vec{l}'_\perp).$$

Below we will construct an equation for $G(\vec{l}_\perp, \vec{l}'_\perp, Y)$ by analyzing the one gluon order-α_s correction

* Effective emission vertex.

First consider corrections to the quark-quark high energy scattering amplitude,

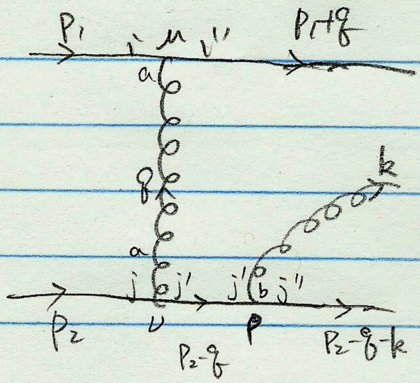




In order to extract the leading-$\ln s$ contribution, we assume $k^+ \ll p^+$, $k^- \ll p^-$

We will perform calculation in $\eta\cdot A = A^+ = 0$ light cone gauge

In this gauge, D and E do not contribute at high energy.

(D) 

$$i\mathcal{M}^{\rho}_{qq\to qqG} = \bar u_{\sigma_1'}(p_1+q)\,(-ig(t^a)_{i'i}\gamma^\mu)\,u_{\sigma_1}(p_1)$$

$$\times G_{\mu\nu}(q)\times \bar u_{\sigma_2'}(p_2-q-k)\,(-ig(t^b)_{j''j'}\gamma^\rho)\,\epsilon^*_\rho(k)$$

$$\times \frac{i(\not p_2-\not q)}{(p_2-q)^2+i\epsilon}\,(-ig(t^a)_{j'j}\gamma^\nu)\,u_{\sigma_2}(p_2).$$

We have $p_1=(P^+,0,\underline{0})$, $p_2=(0,P^-,\underline{0})$.

$P^+\gg k^+$, $P^-\gg k^-$,

$\bar u_{\sigma_1'}(p_1+q)\,\gamma^\mu\,u_{\sigma_1}(p_1) = 2p_1^+\delta_{\sigma_1'\sigma_1}$

$\bar u_{\sigma_2'}(p_2-q-k)\,\gamma^\nu\,u_{\sigma_2}(p_2) = 2p_2^-\delta_{\sigma_2'\sigma_2}$

$(p_2-q)\cdot\gamma = \frac12(p_2-q)^+\gamma^- + \frac12(p_2-q)^-\gamma^+ - (\underline{p_2-q})\,\underline{\gamma} \approx \frac12 p_2^-\gamma^+$

$G_{\mu\nu}(q) = \frac{i}{q^2}\left(g_{\mu\nu} - \frac{\eta_\mu q_\nu+\eta_\nu q_\mu}{\eta\cdot q}\right)$, $\epsilon^{*\mu}_\lambda(k) = \left(0, \frac{2\vec\epsilon_\perp^{\,\lambda}\cdot\vec k}{k^+}, \vec\epsilon_\perp^{\,\lambda}\right)$

We have $(p_1+q)^2=0 \Rightarrow 2p_1\cdot q=0 \Rightarrow \frac12 p_1^+q^- - \underline{p_1}\,\underline{q} = 0(\frac{1}{P^+}) \Rightarrow q^- = 0(\frac{1}{P^+})$

$(p_2-q-k)^2=0 \Rightarrow -2p_2\cdot q - 2p_2\cdot k = 0 \Rightarrow q^+ \simeq k^+ + 0(\frac{1}{P^-})$

$\therefore q^2 = q^+q^- - q_\perp^2 \approx q_\perp^2$ since $q^- = 0(\frac{1}{P^+})$.

For q_μ terms. $\bar u_{\sigma_1'}(p_1+q)\,\not q\,u_{\sigma_1}(p_1) = \bar u_{\sigma_1'}(p_1+q)\,((\not p_1+\not q)-\not p_1)\,u_{\sigma_1}(p_1)$

$= \bar u_{\sigma_1'}(p_1+q)(\not p_1+\not q)u_{\sigma_1}(p_1) - \bar u_{\sigma_1'}(p_1+q)\not p_1 u_{\sigma_1}(p_1)$

$=0$

For η_μ terms. $\bar u_{\sigma_1'}(p_1+q)\,\gamma^\mu\,u_{\sigma_1}(p_1)\,\eta_\mu = \bar u_{\sigma_1'}(p_1+q)\,\gamma^+\,u_{\sigma_1}(p_1) = 2p_1^+\delta_{\sigma_1'\sigma_1}$

For $g_{\mu\nu}$, we need to take g_{+-},

$\bar u_{\sigma_2'}(p_2-q-k)\,\gamma^\rho(\not p_2-\not q)\gamma^-\,u_{\sigma_2}(p_2) = \bar u_{\sigma_2'}(p_2-q-k)\,\gamma^\rho\gamma^+\gamma^-\,u_{\sigma_2}(p_2)\,\frac12 p_2^-$

since $(\gamma^+)^2=0$ ρ has to be "$-$" or "$\perp$"

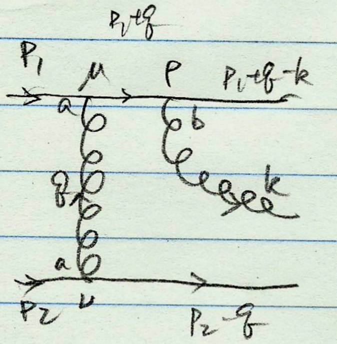
for $\gamma_\perp$, it is suppressed by $\frac{1}{P^-}$

for γ_-, since it contract with $\epsilon_+^*=0$. $\therefore$ do not contribute

For η_μ terms, $\bar u_{\sigma_2'}(p_2-q-k)\,\gamma^\rho(\not p_2-\not q)\not q\,u_{\sigma_2}(p_2) = \bar u_{\sigma_2'}(p_2-q-k)\,\gamma^\rho(\not p_2-\not q)(\not q-\not p_2)u_{\sigma_2}(p_2) = -(p_2-q)^2\,\bar u_{\sigma_2'}(p_2)\gamma^\rho u_{\sigma_2}(p_2)$

$\cancel{\approx}$ $+\,p_2^-q^-\,\bar u_{\sigma_2}(p_2-q-k)\,\gamma^\rho\,u_{\sigma_2}(p_2)$ dominant term will be γ^- again

cancelled be $\epsilon_+^*=0$



$$iM^B_{qq\to qqG} = \bar{U}_{\sigma_2'}(P_2-q)\,(ig\gamma^\nu (t^a)_{j'j})\,U_{\sigma_2}(P_2)$$
$$\times \frac{i}{q^2}\left(g_{\mu\nu} - \frac{\eta_\mu q_\nu + \eta_\nu q_\mu}{\eta\cdot q}\right)$$
$$\times \bar{U}_{\sigma_1}(P_1+q-k)\,(ig\gamma^\rho (t^b)_{i''i'}\,\epsilon^*_{\rho\lambda}(k))\,\frac{\not P_1+\not q}{(P_1+q)^2}$$
$$\times (ig\gamma^\mu (t^a)_{i'i})\,U_\sigma(P_1)$$

$P_1 = (P^+, 0, 0)$

$P_2 = (0, P^-, 0)$

$\epsilon^*_{\lambda\rho}(k) = (0, \frac{\vec{\epsilon}_\perp\cdot\vec{k}_\perp}{k^+}, \vec{\epsilon}_\perp)$

$q^2 \approx -q_\perp^2$

$$= g^3 (t^b t^a)_{i''i}(t^a)_{j'j}\,\bar{U}_{\sigma_2'}(P_2-q)\gamma^\nu U_{\sigma_2}(P_2)\left(-\frac{i}{q^2}\right)$$
$$\times \left(g_{\mu\nu} - \frac{\eta_\mu q_\nu + \eta_\nu q_\mu}{\eta\cdot q}\right)$$
$$\times \bar{U}_{\sigma_1'}(P_1+q-k)\,\epsilon^*_{\rho\lambda}(k)\,\gamma^\rho \frac{\not P_1+\not q}{(P_1+q)^2}\gamma^\mu U_{\sigma_1}(P_1)$$

We have $(P_1+q)\cdot\gamma \approx \frac{1}{2}P^+\gamma^-$, $(P_1+q)^2 = 2P_1\cdot q = P^+q^-$

$$iM^B_{qq\to qqG} = g^3 (t^b t^a)_{i''i}(t^a)_{j'j}\,2P^-\delta_{\sigma_2'\sigma_2}\,\frac{1}{2}\left(-\frac{i}{q^2}\right)$$
$$\times \bar{U}_{\sigma_1'}(P_1+q-k)\,\epsilon^*_{\rho\lambda}(k)\,\gamma^\rho\,\frac{\frac{1}{2}P^+\gamma^-}{P^+q^-}\,\gamma^+ U_{\sigma_1}(P)$$
$$= g^3 (t^b t^a)_{i''i}(t^a)_{j'j}\,P^-\delta_{\sigma_2'\sigma_2}\left(-\frac{i}{q^2}\right) \times \frac{1}{2}\frac{1}{q^-}\epsilon^*_{\rho\lambda}(k)$$
$$\bar{U}_{\sigma_1'}(P_1+q-k)\,\gamma^\rho\gamma^-\gamma^+ U_{\sigma_1}(P_1)$$

$\gamma^+\gamma^- + \gamma^-\gamma^+ = 4 \Rightarrow \gamma^-\gamma^+ = 4 - \gamma^+\gamma^-$

take "+"

suppressed by $\frac{1}{P^+}$

$$= g^3(t^b t^a)_{i''i}(t^a)_{j'j}\,\delta_{\sigma_2'\sigma_2}\,P^-\frac{1}{2}\frac{1}{q^-}\left(-\frac{i}{q^2}\right)\epsilon^*_{\rho\lambda}(k)\,\bar{U}_{\sigma_1'}(P_1+q-k)\gamma^\rho(4-\gamma^+\gamma^-)U_{\sigma_1}(P)$$

take "−"

$$= g^3(t^b t^a)_{i''i}(t^a)_{j'j}\,\delta_{\sigma_2'\sigma_2}\,P^-\frac{1}{2}\frac{1}{q^-}\left(-\frac{i}{q^2}\right)\left(\frac{\vec{\epsilon}_\perp\cdot\vec{k}_\perp}{k^+}\right)2P^+\delta_{\sigma_1'\sigma}\cdot 4$$

$$= 4ig^3(t^b t^a)_{i''i}(t^a)_{j'j}\,\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\,\frac{P^+P^-}{q_\perp^2}\,\frac{1}{q^-}\left(\frac{\vec{\epsilon}_\perp\cdot\vec{k}_\perp}{k^+}\right)$$ $P^+P^- = s.$

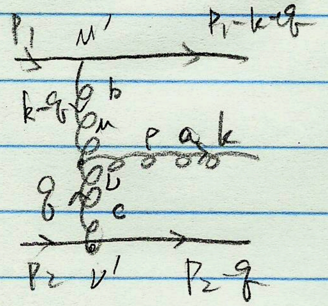
$$= 4ig^3(t^b t^a)_{i''i}(t^a)_{j'j}\,\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\,\frac{s}{q_\perp^2}\,\frac{1}{q^-}\left(\frac{\vec{\epsilon}_\perp\cdot\vec{k}_\perp}{k^+}\right).$$

$q^- \approx k^-$

$k^2 = k^+k^- - k_\perp^2 = 0$

$k^+k^- = k_\perp^2$

$$= 4ig^3(t^b t^a)_{i''i}(t^a)_{j'j}\,\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\,\frac{s}{q_\perp^2}\left(\frac{\vec{\epsilon}_\perp\cdot\vec{k}_\perp}{k_\perp^2}\right)$$



$$i\mathcal{M}^{A}_{qq\to qqG} = \bar{u}_{\sigma_2'}(p_2-q)\left(ig(t^c)_{j'j}\gamma^{\nu'}\right)u_{\sigma_2}(p_2)$$

$$\times \frac{i}{q^2}\left(g_{\nu'\nu} - \frac{\eta_{\nu'}q_\nu + \eta_\nu q_{\nu'}}{\eta\cdot q}\right)\frac{i}{(k-q)^2}\left(g_{\mu'\mu} - \frac{\eta_{\mu'}(k-q)_\mu + \eta_\mu(k-q)_{\mu'}}{\eta\cdot q}\right)$$

$$\times \bar{u}_{\sigma_1'}(p_1-k+q)\left(ig(t^b)_{i'i}\gamma^{\mu'}\right)u_{\sigma_1}(p_1)\,\epsilon^*_{\rho\lambda}(k)$$

$$\times g f^{abc}\left[g^{\rho\mu}(q-2k)^\nu + g^{\mu\nu}(k-2q)^\rho + g^{\nu\rho}(q+k)^\mu\right]$$

$$= g^3 f^{abc}(t^b)_{i'i}(t^c)_{j'j}\,\bar{u}_{\sigma_2'}(p_2-q)\gamma^{\nu'}u_{\sigma_2}(p_2)\frac{1}{q^2(k-q)^2}$$

$$\times \Big(\underset{g_{+-}}{g_{\nu\nu'}} - \underbrace{\cancel{\frac{\eta_\nu q_{\nu'}+\eta_{\nu'}q_\nu}{\eta\cdot q}}}_{O\left(\frac{1}{(p^-)^2}\right)}\Big)\left(g_{\mu\mu'} - \frac{\eta_\mu(k-q)_{\mu'}+\eta_{\mu'}(k-q)_\mu}{\eta\cdot(k-q)}\right)$$

$$\times \bar{u}_{\sigma_1'}(p_1-k+q)\gamma^{\mu'}u_{\sigma_1}(p_1)\,\epsilon^*_{\rho\lambda}(k)$$

$$\times\left[g^{\rho\mu}(q-2k)^\nu + g^{\mu\nu}(k-2q)^\rho + g^{\nu\rho}(q+k)^\mu\right]$$

$$= g^3 f^{abc}(t^b)_{i'i}(t^c)_{j'j}\frac{1}{q^2(k-q)^2}\,2p^-\delta_{\sigma_2'\sigma_2}\overset{g_{+-}}{\frac{1}{2}}\left(g_{\mu\mu'} - \frac{\eta_\mu(k-q)_{\mu'}+\eta_{\mu'}(k-q)_\mu}{\eta\cdot(k-q)}\right)$$

$$\times\left[g^{\rho\mu}(q-2k)^+ + g^{\mu+}(k-2q)^\rho + g^{+\rho}(q+k)^\mu\right]\bar{u}_{\sigma_1'}(p_1-k+q)\gamma^{\mu'}u_{\sigma_1}(p)\,\epsilon^*_{\rho\lambda}(k)$$

$$= g^3 f^{abc}(t^b)_{i'i}(t^c)_{j'j}\,\delta_{\sigma_2'\sigma_2}\frac{1}{q^2(k-q)^2}\,p^-\left(g_{\mu\mu'} - \frac{\eta_\mu(k-q)_{\mu'}+\eta_{\mu'}(k-q)_\mu}{\eta\cdot(k-q)}\right)$$

$$\epsilon^*_{\rho\lambda}(k)\left(g^{\rho\mu}(q-2k)^+ + g^{\mu+}(k-2q)^\rho + g^{+\rho}(q+k)^\mu\right)\bar{u}_{\sigma_1'}(p_1-k+q)\gamma^{\mu'}u_{\sigma_1}(p_1)$$

For $\epsilon^*_{\rho\lambda}(k)\,g^{+\rho}(q+k)^\mu = \epsilon^{+*}_{\lambda}(k)(q+k)^\mu = 0$.

$$g^{\mu+}(k-2q)^\rho\left(g_{\mu\mu'} - \frac{\eta_\mu(k-q)_{\mu'}+\eta_{\mu'}(k-q)_\mu}{\eta\cdot(k-q)}\right) \qquad \boxed{\bar{u}_{\sigma_1'}(p_1-k+q)(\not{k}-\not{q})u_{\sigma_1}(p_1)=0}$$

$$= (k-2q)^\rho\left(g^{+}{}_{\mu'} - \frac{\eta_{\mu'}(k-q)^+}{(k-q)^+}\right) = (k-2q)^\rho\left(g^{+}{}_{\mu'} - \eta_{\mu'}\right) = 0$$

$\therefore$ only $g^{\rho\mu}$ term will contribute.

$$\therefore\ iM^{A}_{qq\to qqg} = g^3 f^{abc} (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_2'\sigma_2} \frac{1}{q^2(k-q)^2} P^- \Big(g_{\mu\nu'} - \frac{\eta_{\nu'}(k-q)_\mu}{(k-q)^+} \Big)$$

$$\epsilon^{*\mu}_\lambda(k)\, ((q-2k)^+)\ 2P^+ \delta_{\sigma_1'\sigma_1}$$

$\eta_+ = g_{+-}\eta^- = 1$

$$= g^3 f^{abc} (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_1'\sigma_1} \delta_{\sigma_2'\sigma_2} \frac{2P^+P^-(q-2k)^+}{q^2(k-q)^2} \Big(g_{\mu +} - \frac{\eta_+ (k-q)_\mu}{(k-q)^+} \Big) \epsilon^{*\mu}_\lambda(k)$$

$$= 2g^3 f^{abc} (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_1'\sigma_1} \delta_{\sigma_2'\sigma_2} \frac{s\,(q-2k)^+}{q^2(k-q)^2} \Big(\cancel{\tfrac{1}{2}\epsilon^{*-}_\lambda(k)} - \cancel{\tfrac{1}{2}\epsilon^{*-}_\lambda(k)} + \frac{\vec{\epsilon}^{\,*}_{\lambda\perp}\cdot(\vec{k}_\perp - \vec{q}_\perp)}{(k-q)^+} \Big)$$

$$= 2g^3 f^{abc} (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_1'\sigma_1} \delta_{\sigma_2'\sigma_2} \frac{s\,(q-2k)^+}{q^2(k-q)^2} \frac{\vec{\epsilon}^{\,*}_{\lambda\perp}\cdot(\vec{k}_\perp - \vec{q}_\perp)}{(k-q)^+}$$

Since $q^+ \approx O(\frac{1}{P^-}) \ll k^+$,

$$= -4g^3 f^{abc} (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_1'\sigma_1} \delta_{\sigma_2'\sigma_2} \frac{s}{q^2(k-q)^2} \vec{\epsilon}^{\,*}_{\lambda\perp}\cdot(\vec{k}_\perp - \vec{q}_\perp)$$

$q^2 \approx -q_\perp^2$, $\quad (k-q)^2 = (k_+ - q_+)(k_- - q_-) - (\vec{k}_\perp - \vec{q}_\perp)^2$

$q_- \approx k_- + O(\frac{1}{P^+})\ \therefore\ k_- - q_- = O(\frac{1}{P^+})\ \therefore\ (k-q)^2 = -(\vec{k}_\perp - \vec{q}_\perp)^2$

$$\therefore\ = -4g^3 f^{abc} (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_1'\sigma_1} \delta_{\sigma_2'\sigma_2} \frac{s}{q_\perp^2(\vec{k}_\perp - \vec{q}_\perp)^2} \vec{\epsilon}^{\,*}_{\lambda\perp}\cdot(\vec{k}_\perp - \vec{q}_\perp).$$

$$\therefore\ M^{A}_{qq\to qqg} = 4ig^3 f^{abc} (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_1'\sigma_1} \delta_{\sigma_2'\sigma_2} \frac{s}{q_\perp^2(\vec{k}_\perp - \vec{q}_\perp)^2} \vec{\epsilon}^{\,*}_{\lambda\perp}\cdot(\vec{k}_\perp - \vec{q}_\perp)$$

$$\therefore\ M^{B+C}_{qq\to qqG} = -4ig^3 f^{abc} (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_1'\sigma_1} \delta_{\sigma_2'\sigma_2} \frac{s}{q_\perp^2} \frac{\epsilon^{*}_{\lambda\perp}\cdot \vec{k}_\perp}{k_\perp^2}$$

$$\therefore\ M_{qq\to qqG} = 4ig^3 f^{abc} (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_1'\sigma_1} \delta_{\sigma_2'\sigma_2} \frac{s}{q_\perp^2(\vec{k}_\perp - \vec{q}_\perp)^2} \vec{\epsilon}^{\,*}_{\lambda\perp}\cdot\Big(\vec{k}_\perp - \vec{q}_\perp - \frac{(\vec{k}_\perp - \vec{q}_\perp)^2}{k_\perp^2} \vec{k}_\perp \Big)$$

$$= 2ig^2 (t^b)_{i'i} (t^c)_{j'j}\, \delta_{\sigma_1'\sigma_1} \delta_{\sigma_2'\sigma_2} \frac{s}{q_\perp^2(\vec{k}_\perp - \vec{q}_\perp)^2} \vec{\epsilon}^{\,\lambda *}_\perp \cdot \vec{\Gamma}^{abc}_\perp$$

$$\vec{\Gamma}^{abc}_\perp = 2g f^{abc} \Big[\vec{k}_\perp - \vec{q}_\perp - \frac{(\vec{k}_\perp - \vec{q}_\perp)}{k_\perp^2} \vec{k}_\perp \Big]$$

Adding all the diagram,

$$\mathcal{M}_{qq\to qqG} = 2ig^2 (t^b)_{i'i}(t^c)_{j'j}\,\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\frac{s}{q_\perp^2(\vec{k}_\perp-\vec{q}_\perp)^2}\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{\Gamma}_\perp^{abc}$$

$$\vec{\Gamma}_\perp^{abc} = 2gf^{abc}\left[\vec{k}_\perp-\vec{q}_\perp-\frac{(\vec{k}_\perp-\vec{q}_\perp)^2}{k_\perp^2}\vec{k}_\perp\right]$$

By squaring the amplitude,

$$\sigma_{qq\to qqG} = \frac{2\alpha_s^3 C_F}{\pi^2}\int\frac{d^2k_\perp\, d^2q_\perp}{k_\perp^2 q_\perp^2(\vec{k}_\perp-\vec{q}_\perp)^2}\int_{k_\perp^2/p^-}^{p^+}\frac{dk^+}{k^+}$$

We define the rapidity of the gluon $y = \ln\frac{p^-}{k^-}$

$$\sigma_{qq\to qqG} = \frac{2\alpha_s^3 C_F}{\pi^2}\int\frac{d^2k_\perp\, d^2q_\perp}{k_\perp^2 q_\perp^2(\vec{k}_\perp-\vec{q}_\perp)^2}\int_0^Y dy$$

Here $Y = \ln\frac{s}{k_\perp^2}$ is the total rapidity interval between the colliding quark.

For onium + onium scattering,

$$\hat{\sigma}_{1,\,real}^{onium+onium} = \frac{2\alpha_s^2 C_F}{N_c}\int\frac{d^2l_\perp\, d^2l_\perp'}{l_\perp^2\, l_\perp'^2}\left(2-e^{-i\vec{l}_\perp\cdot\vec{x}_{1\perp}}-e^{i\vec{l}_\perp\cdot\vec{x}_{1\perp}}\right)$$

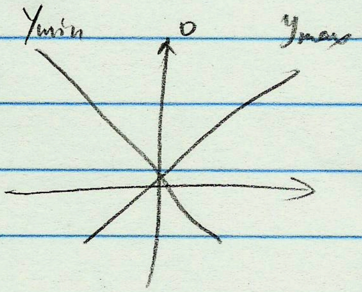
$$\times\left(2-e^{-i\vec{l}_\perp'\cdot\vec{x}_{2\perp}}-e^{i\vec{l}_\perp'\cdot\vec{x}_{2\perp}}\right)G_1^{real}(\vec{l}_\perp,\vec{l}_\perp',Y).$$

$$G_1^{real}(\vec{l}_\perp,\vec{l}_\perp',Y) = \frac{\alpha_s N_c}{\pi^2}Y\frac{1}{(\vec{l}_\perp-\vec{l}_\perp')^2}$$

$$\therefore G(\vec{l}_\perp,\vec{l}_\perp',Y) = G_0(\vec{l}_\perp,\vec{l}_\perp') + G_1^{real}(\vec{l}_\perp,\vec{l}_\perp',Y)+\cdots$$

$$= G_0(\vec{l}_\perp,\vec{l}_\perp') + \frac{\alpha_s N_c}{\pi^2}\int_0^Y dy\int\frac{d^2q_\perp}{(\vec{l}_\perp-\vec{q}_\perp)^2}G_0(\vec{q}_\perp,\vec{l}_\perp')+\cdots$$

Rapidity



For a given k^μ,

$$y = \frac{1}{2}\ln\frac{k^+}{k^-}$$

since $k^2 = 0 = k^+k^- - k_\perp^2 \Rightarrow k^+k^- = k_\perp^2$

$$\therefore y = \frac{1}{2}\ln\frac{k^+}{k^-} = \frac{1}{2}\ln\left(\frac{k^+}{k_\perp}\right)^2 = \ln\frac{k^+}{k_\perp} = \ln\frac{k_\perp}{k^-}$$

Assuming the largest momentum in our system is P^+ and P^-, then

$$y_{max} = \ln\frac{P^+}{k_\perp} \qquad y_{min} = \ln\frac{k_\perp}{P^-}$$

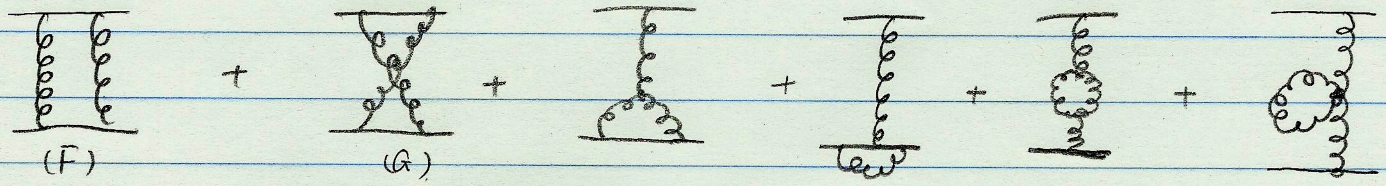
Then the total range of rapidity it covers will be

$$Y = y_{max} - y_{min} = \ln\frac{P^+P^-}{(k_\perp)^2} = \ln\left(\frac{s}{k_\perp^2}\right).$$

And now if we define $y=0$ at one onium, say at y_{min}, a particle with momentum k will have rapidity,

$$y = \ln\frac{k_\perp}{k^-} - y_{min} = \ln\frac{k_\perp}{k^-} - \ln\frac{k_\perp}{P^-} = \ln\frac{P^-}{k^-}$$

Virtual corrections and reggeized gluons.



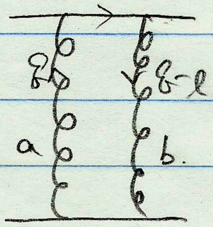
Our goal is to find the order-α_s^2 amplitude in the above diagram.

We will follow double-substracted dispersion relation for scattering amplitude

$$\mathcal{M}(s,t) = \mathcal{M}(s=0,t) + s\,\partial_s \mathcal{M}(s=0,t)$$
$$+ \frac{s^2}{\pi}\left[\int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}_s \mathcal{M}(s',t)}{s'^2(s'-s)} + \int_{4m^2-t}^{\infty} du' \frac{\mathrm{Im}_u \mathcal{M}(u',t)}{(4m^2-t-u')^2(u'-u)} \right]$$

In order to find the order-α_s^2 amplitude, we need the diagrams that have an imaginary part. Thus, only the first two will contribute. Using the optical theorem,

$$\mathrm{Im}\,\mathcal{M}^F_{qq\to qq}(\text{forward}) = s\,\sigma^0_{qq\to qq}.$$



$$\therefore \mathrm{Im}\,\mathcal{M}^F_{qq\to qq} = \int \frac{d^2 q_\perp}{(4\pi)^2 s} \sum \mathcal{M}^0_{qq\to qq}(\vec{q}_\perp) \left(\mathcal{M}^0_{qq\to qq}(\vec{q}_\perp - \vec{l}_\perp) \right)^*$$

In covariant gauge

$$= 4\alpha_s^2 (t^b t^a)_{i'i} (t^b t^a)_{j'j}\, \delta_{\sigma_1\sigma_1'} \delta_{\sigma_2\sigma_2'} \int \frac{d^2 q_\perp}{q_\perp^2 (\vec{q}_\perp - \vec{l}_\perp)^2}\, s.$$

The imaginary part of $\mathcal{M}^G_{qq\to qq}$ is obtained from $\mathrm{Im}\,\mathcal{M}^F_{qq\to qq}$ by interchange s and u, and color indices a and b along the quark line

$$\mathrm{Im}\,\mathcal{M}^G_{qq\to qq} = 4\alpha_s^2 (t^b t^a)_{i'i} (t^a t^b)_{j'j}\, \delta_{\sigma_1\sigma_1'} \delta_{\sigma_2\sigma_2'} \int \frac{d^2 q_\perp}{q_\perp^2 (\vec{q}_\perp - \vec{l}_\perp)^2}\, u.$$

Thus, we have

$$\mathcal{M}'_{qq\to qq}(s, t=-l_\perp^2) = \mathcal{M}'_{qq}(s=0,t) + s\,\partial_s \mathcal{M}'_{qq\to qq}(s=0,t) + \frac{4\alpha_s^2 s^2}{\pi} \delta_{\sigma_1\sigma_1'} \delta_{\sigma_2\sigma_2'}$$
$$\times \int \frac{d^2 q_\perp}{q_\perp^2 (\vec{q}_\perp - \vec{l}_\perp)^2} (t^b t^a)_{i'i} \left[(t^b t^a)_{j'j} \int_{4m^2}^{+\infty} ds' \frac{1}{s'(s'-s)} + (t^a t^b)_{j'j} \int_{4m^2-t}^{\infty} du' \frac{u'}{(4m^2-t-u')^2 (u'-u)} \right]$$

We require the high energy asymptotics of the amplitude $M'_{qq\to qq}(s,t)$

$$M'_{qq\to qq}(s,t=-l_\perp^2) = -\frac{4\alpha_s^2 s}{\pi}\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}\int\frac{d^2q_\perp}{q_\perp^2(\vec{q}_\perp-\vec{l}_\perp)^2}(t^b t^a)_{i'i}$$

$$\times\left[(t^b t^a)_{j'j}\ln\frac{s}{4m^2} - (t^a t^b)_{j'j}\ln\left(\frac{-s}{4m^2-t}\right)\right]$$

$$= \frac{2\alpha_s^2 N_c s}{\pi}(t^a)_{i'i}(t^a)_{j'j}\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}\int\frac{d^2q_\perp}{q_\perp^2(\vec{q}_\perp-\vec{l}_\perp)^2}Y$$

$$= M^0_{qq\to qq}(\vec{l}_\perp)\,\omega_G(l_\perp)\,Y$$

where we have defined the gluon Regge trajectory.

$$\omega_G(l_\perp) = -\frac{\alpha_s N_c}{4\pi^2}\int d^2q_\perp\frac{l_\perp^2}{q_\perp^2(\vec{q}_\perp-\vec{l}_\perp)^2}$$

One can replace the gluon propagator

$$\frac{ig_{\mu\nu}}{l_\perp^2}\to\frac{ig_{\mu\nu}}{l_\perp^2}e^{\omega_G(l_\perp)Y}\sim\frac{ig_{\mu\nu}}{l_\perp^2}s^{\omega_G(l_\perp)}$$

This is effective describe the exchange of a particle with spin $j=1+\omega_G(l_\perp)$
We refer to this "quasi-particle" as a reggeized gluon.

To find the order-α_s virtual correction, we have to consider the interference between the lowest-order amplitude

$$M'_{qq\to qq}(M^0_{qq\to qq})^* + M^0_{qq\to qq}(M'_{qq\to qq})^* = |M^0_{qq\to qq}|^2\,2\omega_G(l_\perp)Y.$$

The correction contributions are,

$$G_1^{virtual}(\vec{l}_\perp,\vec{l}_\perp',Y) = G_0(\vec{l}_\perp,\vec{l}_\perp')\,2\omega_G(l_\perp)Y$$

$\therefore$ The complete order-α_s corrections are

$$G(\vec{l}_\perp,\vec{l}_\perp',Y) = G_0(\vec{l}_\perp,\vec{l}_\perp') + \frac{\alpha_s N_c}{\pi^2}\int_0^Y dy\int\frac{d^2q_\perp}{(\vec{l}_\perp-\vec{q}_\perp)^2}$$

$$\times\left[G_0(\vec{q}_\perp,\vec{l}_\perp') - \frac{l_\perp^2}{2q_\perp^2}G_0(\vec{l}_\perp,\vec{l}_\perp')\right].$$

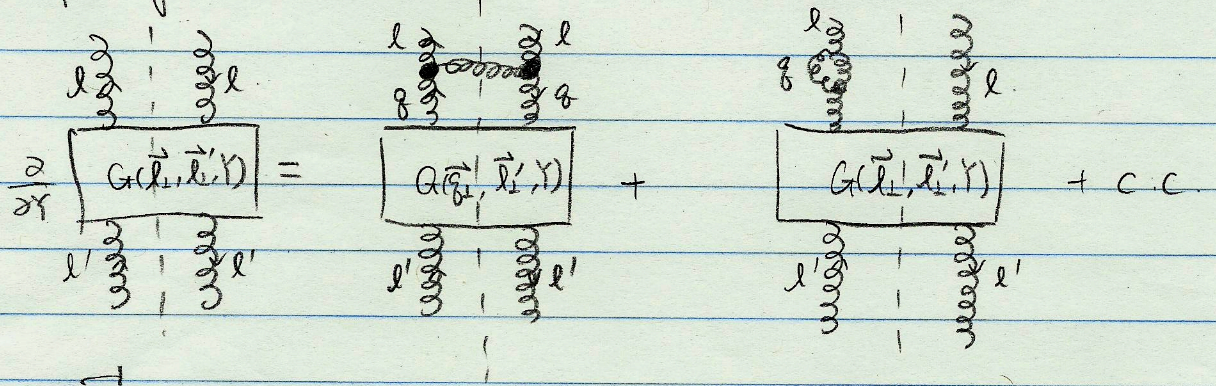
BFKL equation.

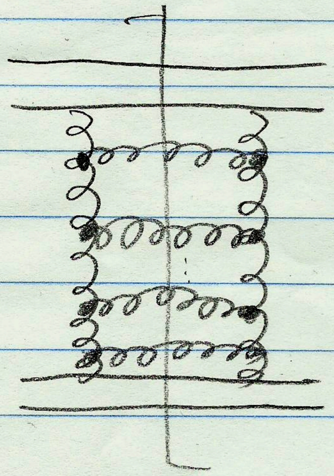
$$\frac{\partial G(\vec{l}_\perp, \vec{l}'_\perp, Y)}{\partial Y} = \frac{\alpha_s N_c}{\pi^2}\int \frac{d^2 q_\perp}{(\vec{l}_\perp - \vec{q}_\perp)^2}\left[G(\vec{q}_\perp, \vec{l}'_\perp, Y) - \frac{l_\perp^2}{2 q_\perp^2} G(\vec{l}_\perp, \vec{l}'_\perp, Y)\right]$$

With initial condition.

$$G(\vec{l}_\perp, \vec{l}'_\perp, Y=0) = \delta^{(2)}(\vec{l}_\perp - \vec{l}'_\perp).$$

This equation resums all leading-logs correction to Born-level onium-onium scattering amplitude. $G(\vec{l}_\perp, \vec{l}'_\perp, Y)$ is called the Green function of the BFKL equation: it describes the propagation of two t-channel gluons over the rapidity interval Y.





There are two main ingredients that describe high energy scattering in leading-log approximation:

* the reggeized gluon
* the new effective (Lipatov) vertex

Compare to DGLAP evolution.

* DGLAP ladder would include quarks. But BFKL does not.
* DGLAP are strongly order in partons transverse momenta, BFKL is the opposite. parton's longitudinal momenta are strongly ordered.

For BFKL, if the momenta of the gluons are labeled $k_1^\mu, k_2^\mu \ldots$

$$P^+ \gg k_1^+ \gg k_2^+ \gg \cdots \gg k_n^+$$
$$k_1^- \ll k_2^- \ll \cdots \ll k_n^- \ll P^-$$
$$k_{1\perp} \sim k_{2\perp} \sim \cdots \sim k_{n\perp}$$

The kinematics is known as the multi-Regge kinematics. In terms of rapidities that

$$Y \gg y_1 \gg y_2 \gg \cdots \gg y_n \gg 0.$$

We see that the multi-Regge kinematics corresponds to the situation where the produced gluons uniformly cover the whole available rapidity interval. The BFKL approach gives us the possibility to calculate the exclusive production cross section for any given number of gluons in the multi-Regge kinematics.

* Solution of the BFKL equation

To find a general solution of BFKL equation, we need to find eigenfunctions of its integral kernel K_{BFKL} defined by,

$$\int d^2q_\perp \, K_{BFKL}(l,q)\, f(\vec{q}_\perp) = \frac{1}{\pi}\int \frac{d^2q_\perp}{(\vec{l}_\perp - \vec{q}_\perp)^2}\left[f(\vec{q}_\perp) - \frac{l_\perp^2}{2q_\perp^2} f(\vec{l}_\perp)\right]$$

for an arbitrary function $f(\vec{q}_\perp)$. The eigenfunctions have the form,

$$l_\perp^{2(\gamma-1)} e^{in\phi_l}$$

with γ an arbitrary complex number. ϕ_l is angle between the vector $\vec{l}_\perp$ and some chosen axis in the transverse plane and n is an integer.

check: $\int d^2q_\perp \, K_{BFKL}(l,q)\, q_\perp^{2(\gamma-1)} e^{in\phi_q} = \frac{1}{\pi}\int \frac{d^2q_\perp}{(\vec{l}_\perp - \vec{q}_\perp)^2}\left[q_\perp^{2(\gamma-1)} e^{in\phi_q} - \frac{l_\perp^2}{2q_\perp^2} l_\perp^{2(\gamma-1)} e^{in\phi_l}\right]$

$$\frac{1}{q_\perp^2(\vec{l}_\perp - \vec{q}_\perp)^2} = \frac{1}{q_\perp^2\left(q_\perp^2 + (\vec{l}_\perp - \vec{q}_\perp)^2\right)} + \frac{1}{(\vec{l}_\perp - \vec{q}_\perp)^2\left[q_\perp^2 + (\vec{l}_\perp - \vec{q}_\perp)^2\right]}$$

$$\int \frac{d^2 q_\perp}{q_\perp^2 (\vec{l}_\perp - \vec{q}_\perp)^2} = 2 \int \frac{d^2 q_\perp}{(\vec{l}_\perp - \vec{q}_\perp)^2 [q_\perp^2 + (\vec{l}_\perp - \vec{q}_\perp)^2]}$$

$$\int d^2 q_\perp \, K_{BFKL}(l, q) \, q_\perp^{2(\gamma-1)} e^{in\phi_q}$$

$$= \frac{1}{\pi} \int d^2 q_\perp \left\{ \frac{q_\perp^{2(\gamma-1)} e^{in\phi_q}}{(\vec{l}_\perp - \vec{q}_\perp)^2} - \frac{l_\perp^{2\gamma} e^{in\phi_l}}{q_\perp^2} \left(\frac{1}{(\vec{l}_\perp - \vec{q}_\perp)^2} - \frac{1}{q_\perp^2 + (\vec{l}_\perp - \vec{q}_\perp)^2} \right) \right\}$$

$$\therefore \int d^2 q_\perp \, K_{BFKL}(l, q) \, q_\perp^{2(\gamma-1)} e^{in\phi_q} = \chi(n, \gamma) \, l_\perp^{2(\gamma-1)} e^{in\phi_l}.$$

where

$$\chi(n, \gamma) = \int_0^{+\infty} dt \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi_q}{1 + t - 2\sqrt{t}\cos(\phi_q - \phi_l)} t^{\gamma - 1} e^{in(\phi_q - \phi_l)} - \frac{1}{t} \left(\frac{1}{|t-1|} - \frac{1}{\sqrt{4t^2+1}} \right) \right]$$

with $t = q_\perp^2 / l_\perp^2$.

So $l_\perp^{2(\gamma-1)} e^{in\phi_l}$ is indeed an eigenfunction of BFKL kernel, with $\chi(n, \gamma)$ the corresponding eigenvalue.

$$\chi(n, \gamma) = 2\psi(1) - \psi\left(\gamma + \frac{|n|}{2}\right) - \psi\left(1 - \gamma + \frac{|n|}{2}\right)$$

where

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \int_0^1 dt \, \frac{t^{z-1} - 1}{t - 1} + \psi(1), \quad (\mathrm{Re}\, z > 0)$$

Expanding the general solution of the BFKL equation over the eigenfunctions of the BFKL kernel, $G(\vec{l}_\perp, \vec{l}_\perp', Y) = G(\vec{l}_\perp', \vec{l}_\perp, Y)$

$$G(\vec{l}_\perp, \vec{l}_\perp', Y) = \sum_{n=-\infty}^{+\infty} \int_{a-i\infty}^{a+i\infty} \frac{d\gamma}{2\pi i} C_{n,\gamma}(Y) \, l_\perp^{2(\gamma-1)} \, l_\perp'^{\,2(\gamma^*-1)} e^{in(\phi - \phi')}$$

where the $C_{n,\gamma}(Y)$ are some unknown function.

$$C_{n,\gamma}(Y) = C^0_{n,\gamma} \exp\left\{ \frac{\alpha_s N_c}{\pi} \chi(n, \gamma) Y \right\}$$

where the coefficient $C^0_{n,\gamma}$ is fixed by the initial condition,

$$C^0_{n,\gamma} = \frac{1}{\pi}$$

$$\therefore G(\vec{l}_\perp, \vec{l}_\perp', Y) = \sum_{n=-\infty}^{n=+\infty} \int_{1/2-i\infty}^{1/2+i\infty} \frac{d\gamma}{2\pi i} \exp\left\{ \frac{\alpha_s N_c}{\pi} \chi(n, \gamma) Y \right\} l_\perp^{2(\gamma-1)} \, l_\perp'^{\,2(\gamma^*-1)} e^{in(\phi - \phi')}$$

Define $\gamma = a + i\nu = \frac{1}{2} + i\nu$ (ν is real)

$\therefore$ $$G(\vec{l}_\perp, \vec{l}_\perp', Y) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi^2} \exp\left[\frac{\alpha_s N_c}{\pi} \chi(n,\nu) Y\right] l_\perp^{-1+2i\nu} l_\perp'^{-1-2i\nu} e^{in(\phi-\phi')}$$

Where $$\chi(n,\nu) = 2\psi(1) - \psi\left(\frac{1+|n|}{2} + i\nu\right) - \psi\left(\frac{1+|n|}{2} - i\nu\right)$$

The exact analytic solution is not feasible.

Diffusion Approximation

Consider $l_\perp \sim l_\perp'$, we will use saddle point method to evaluate the ν-integral.

At $\nu = 0$, the dominant contribution to the amplitude is given by the $n=0$ term in the sum. So we will only keep $n=0$.

$$G(\vec{l}_\perp, \vec{l}_\perp', Y) = \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi^2 l_\perp l_\perp'} \exp\left\{\bar{\alpha}_s \chi(0,\nu) Y + 2i\nu \ln\frac{l_\perp}{l_\perp'}\right\}$$

$\chi(0,\nu) \approx 4\ln 2 - 14\zeta(3)\nu^2$ around the saddle point

$\therefore$ $$G(\vec{l}_\perp, \vec{l}_\perp', Y) \approx \frac{1}{2\pi^2 l_\perp l_\perp'} \sqrt{\frac{\pi}{14\zeta(3)\bar{\alpha}_s Y}} \exp\left\{(\alpha_p - 1) Y - \frac{\ln^2(l_\perp/l_\perp')}{14\zeta(3)\bar{\alpha}_s Y}\right\}.$$

$$\alpha_p - 1 = \frac{4\alpha_s N_c}{\pi} \ln 2$$

The essential feature of the solution is that the cross section mediated by BFKL ladder exchange grow as a power of the energy:

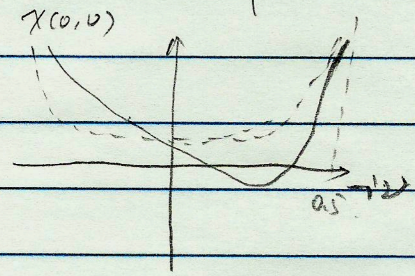
$$\sigma \sim e^{(\alpha_p - 1) Y} \sim s^{\alpha_p - 1}$$

BFKL evolution modifies the energy independent Low-Nussinov pomeron, which makes the perturbative pomeron intercept $\alpha_p > 1$.

The numerical value of the BFKL intercept is rather large. for $\alpha_s = 0.3$ one gets $\alpha_p - 1 \approx 0.79$.

Double logarithmic approximation

Consider $l_\perp \gg l_\perp'$. Now $\ln(l_\perp/l_\perp')$ is large and it may affect the location of the saddle point of the ν-integral.



same behavior for other n.

Large $l_\perp/l_\perp'$ shift the saddle point in the imaginary ν direction moving closer to the singularity. $\nu = i(|n|+1)/2$.

We see that with $|n|>0$,

$$\frac{1}{l_\perp^2}\left(\frac{l_\perp'^2}{l_\perp^2}\right)^{|n|} \quad \text{are suppressed by powers of } \frac{l_\perp'^2}{l_\perp^2} \ll 1$$

compared with $n=0$ term.

Expanding the $n=0$ eigenvalue of BFKL kernel near $\nu = i/2$,

$$\chi(0,\nu) \approx -\frac{i}{\nu - i/2}$$

and the saddle point of the integral is then given by

$$\nu_{DLA} = \frac{i}{2} - i\sqrt{\frac{\bar\alpha_s Y}{\ln(l_\perp^2/l_\perp'^2)}}$$

$$\therefore G(\vec{l}_\perp, \vec{l}_\perp', Y) \approx \frac{1}{2\pi^{3/2} l_\perp^2}\frac{(\bar\alpha_s Y)^{1/4}}{\ln^{3/4}(l_\perp^2/l_\perp'^2)} \exp\left\{2\sqrt{\bar\alpha_s Y \ln(l_\perp^2/l_\perp'^2)}\right\}$$

We see the DLA limit is indeed the same as obtained from DGLAP.

We can now think of BFKL evolution as a property of the hadronic light cone wave function. (will show in Chapter 4)

Define the unintegrated gluon distribution.

$$\phi(x_{Bj}, k_\perp^2) = \frac{\alpha_s C_F}{\pi}\int d^2x_\perp \int_0^1 dz\, |\psi(\vec{x}_\perp, z)|^2$$

$$\times \int \frac{d^2 l_\perp}{l_\perp^2}\left(2 - e^{-i\vec{l}_\perp\cdot\vec{x}_\perp} - e^{i\vec{l}_\perp\cdot\vec{x}_\perp}\right) G(\vec{k}_\perp, \vec{l}_\perp, Y=\ln\frac{1}{x_{Bj}})$$

One can show that in the small-x LLA,

$$\phi(x, Q^2) = \frac{\partial x G(x, Q^2)}{\partial Q^2}$$

☆ This implies that $\phi(x, k_\perp^2)$ counts the number of partons in a hadron at a given value of $k_\perp$.

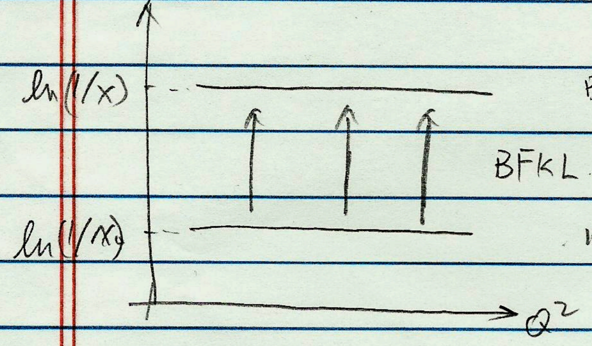
$\phi(x_{Bj}, k_\perp^2)$ obeys the same BFKL evolution equation.

$$\frac{\partial \phi(x, k_\perp^2)}{\partial \ln(1/x)} = \frac{\alpha_s N_c}{\pi^2} \int \frac{d^2 q_\perp}{(\vec{k}_\perp - \vec{q}_\perp)^2} \left(\phi(x, q_\perp^2) - \frac{k_\perp^2}{2 q_\perp^2} \phi(x, k_\perp^2) \right)$$

By analogy, we have

$$\phi(x, k_\perp^2) = \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} C_\nu \exp\left[\frac{\alpha_s N_c}{\pi} \chi(0,\nu) \ln\frac{1}{x}\right] k_\perp^{-1+2i\nu} \Lambda^{-1-2i\nu}$$

With Λ some typical transverse momentum scale characterizing the onium (e.g. the inverse size of the onium). and C_ν an known function determined by the initial condition at $x = x_0$



Given the initial unintegrated gluon contribution $\phi(x_0, k_\perp^2)$, one can find C_ν and then one obtains a unintegrated gluon distribution at other values of x

In the diffusion approximation, $k_\perp \sim \Lambda \gg \Lambda_{QCD}$,

$$\phi(x, k_\perp^2) \approx \frac{c_0}{2\pi} \frac{1}{k_\perp \Lambda} \sqrt{\frac{\pi}{14\zeta(3)\, \bar{\alpha}_s \ln(1/x)}} \left(\frac{1}{x}\right)^{\alpha_P - 1} \exp\left\{-\frac{\ln^2(k_\perp/\Lambda)}{14\zeta(3)\, \bar{\alpha}_s \ln(1/x)}\right\}$$

It grows as a power of $1/x$.

$$\phi(x, k_\perp^2) \sim \left(\frac{1}{x}\right)^{\alpha_P - 1}$$

The small-x growth of gluon distribution is much faster by BFKL than DGLAP.

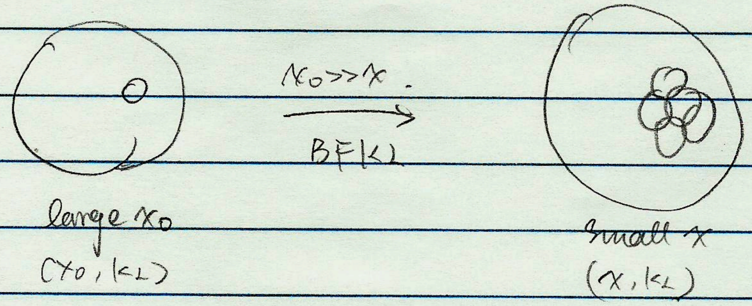
For BFKL evolution:

$$x_1 \ll x_2 \ll \cdots \ll x_n \ll 1$$

The parton n, moves in the light cone minus direction.

Since there is no constrain on transverse momentum, the typical transverse size $x_\perp \sim \frac{1}{k_\perp}$

$$\therefore \ x_{1\perp} \sim x_{2\perp} \sim \cdots \sim x_{n\perp}$$



As x decreases, the gluons have a larger typical longitudinal spread.

Since the gluon increases and the transverse size of each of them is roughly the same, BFKL evolution will achieve high parton density inside hadron.

Problems of BFKL evolution: unitarity and diffusion

* Leading-logrithmic BFKL violates unitarity

Froissart - Martin Bound.

The total cross section for the scattering of the point particle from a disk of radius R is limited by

$$\sigma_{tot} \leq 2\pi R^2$$

The total cross section can be twice as large as the geometric cross section area. $\sigma_{el} = \sigma_{inel} = \pi R^2 \Rightarrow \sigma_{tot} = 2\pi R^2$

Consider Hadron-Hadron scattering at impact parameter b, and b is larger than the black disk limit. And we assume the interaction between hadrons is accomplished through an exchange of one or several particles, $\sigma \sim s^{\Delta}$. Δ is some positive number

At the same time the strength of the interaction should fall off as $\sim e^{-2m_\pi b}$. We thus have a probability p.

$$p \sim s^{\Delta} e^{-2m_\pi b}$$

The upper limit on the radius of the black disk can be determined by requiring $p \approx 1$,

$$s^{\Delta} e^{-2m_\pi b^*} = 1$$

$$\therefore R = b^* \sim \frac{\Delta}{2m_\pi} \ln s$$

$$\therefore \boxed{\sigma_{tot} \leq 2\pi R = \frac{\pi\Delta^2}{2m_\pi^2} \ln^2 s}$$

$\therefore$ The total cross section in QCD can not grow faster than the logarithm of energy squared

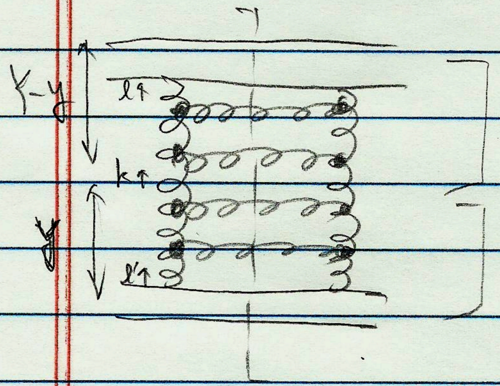
From BFKL, $\sigma_{tot}^{BFKL} \sim s^{\alpha_P - 1}$ which violate the F-M Bound.

$\sigma_{tot}^{DLA\ BFKL} \sim \exp\left[2\sqrt{\bar{\alpha}_s \ln s \ln(l_\perp^2/l_\perp'^2)}\right]$. also violate

DGLAP evolution also violate unitarity.

Diffusion into the infrared

If we look "inside" the BFKL ladder. Let us pick a gluon in the ladder sufficiently ~~far~~ from the ends.



$G(\vec{l}_\perp, \vec{k}_\perp, Y-y)$

$G(\vec{k}_\perp, \vec{l}_\perp', y)$.

The ladder can be splitted into two ladders. The BFKL Green function can be written as a convolution of two BFKL Green function

$$G(\vec{l}_\perp, \vec{l}_\perp', Y) = \int d^2k_\perp\, G(\vec{l}_\perp, \vec{k}_\perp, Y-y)\, G(\vec{k}_\perp, \vec{l}_\perp', y)$$

The $k_\perp$-distribution $\frac{dn}{d^2k_\perp}$ for the t-channel gluon in the diffusion approximation is proportional to

$$\frac{dn}{d\ln k_\perp^2} = k_\perp^2\, G(\vec{l}_\perp, \vec{k}_\perp, Y-y)\, G(\vec{k}_\perp, \vec{l}_\perp', y)$$

$$\sim \exp\left[-\frac{\ln^2(l_\perp/k_\perp)}{14\zeta(3)\bar{\alpha}_s(Y-y)} - \frac{\ln^2(k_\perp/l_\perp')}{14\zeta(3)\bar{\alpha}_s y}\right]$$

The distribution is Gaussian in $\ln k_\perp^2$. and the width depend on rapidity. which is maximized at $y = Y/2$. This means that $k_\perp$ deviate more and more from the original momenta $l_\perp$ and $l_\perp'$. So for $l_\perp, l_\perp' \gg \Lambda_{QCD}$, at $y = Y/2$. (taking $l_\perp = l_\perp'$)

$$l_\perp e^{-\#\sqrt{Y/2}} \lesssim k_\perp \lesssim l_\perp e^{\#\sqrt{Y/2}}$$

as Y increase, $k_\perp$ will diffuse to Λ_{QCD} and becomes non-perturbative

Scattering Cross section.

For an incoming state $|i\rangle$. The final state $|f\rangle = \hat{S}|i\rangle$.

$$= |i\rangle + (\hat{S} - \mathbb{1})|i\rangle$$

The total cross section

$$\sigma_{tot} = |(\hat{S} - \mathbb{1})|i\rangle|^2$$

$$= \langle i|(2 - \hat{S}^+ - \hat{S})|i\rangle \qquad S = \langle i|\hat{S}|i\rangle$$

$$= 2 - 2\mathrm{Re}(S)$$

$$= \int d^2b \, (2 - 2\mathrm{Re}\, S(\vec{b}))$$

$$= 2\int d^2b \, (1 - \mathrm{Re}\, S(\vec{b}))$$

$$\sigma_{el} = |\langle i|(\hat{S} - \mathbb{1})|i\rangle|^2$$

$$= |S - 1|^2$$

$$= |S|^2 - 2\mathrm{Re}\, S + 1$$

$$= \int d^2b \, (|S|^2 - 2\mathrm{Re}(S) + 1)$$

$\therefore \sigma_{tot} \geq \sigma_{el}$

$$\Rightarrow \quad 2(1 - \mathrm{Re}\, S) \geqslant |S|^2 - 2\mathrm{Re}\, S + 1$$

$$\therefore \quad |S|^2 \leq 1$$

If we take $S = 1$, $\sigma_{tot} = \sigma_{el} = 0$. (no interaction)

If we take $S = 0$, (in the high energy limit)

$$\begin{cases} \sigma_{tot} = 2\int d^2b \cdot 1 = 2\pi R^2 \\ \sigma_{el} = \int d^2b = \pi R^2 \\ \sigma_{inel} = \pi R^2. \end{cases}$$

If we take $S = -1$ (in the low energy limit)

$$\begin{cases} \sigma_{tot} = 4\pi R^2 \\ \sigma_{el} = 4\pi R^2 \\ \sigma_{inel} = 0 \end{cases}$$

3.4. The non-linear Gribov-Levin-Ryskin and Muller-Qiu evolution equation.

* Physical picture of parton saturation

Consider onium-onium scattering, the total cross section is then

$$\sigma_{tot}^{onium+onium} \sim \alpha_s^2 \, x_{1\perp} x_{2\perp} \, e^{(\alpha_p - 1)Y}$$

We know $\sigma_{tot}^{onium+onium} \le 2\pi R^2$, with the radius grows logarithmically with energy. R is of the order of a typical hadronic radius $R \simeq r_h$. Let us assume that our onium-onium scattering models a DIS event. $x_{2\perp} \approx r_h$ $x_{1\perp} \approx 1/Q$

$$\alpha_s^2 \frac{r_h}{Q} e^{(\alpha_p - 1)Y} \le 2\pi r_h^2$$

The equality is reached at saturation scale $Q = Q_s$ given by

$$Q_s \sim \alpha_s^2 \Lambda_{QCD} e^{(\alpha_p - 1)Y} = \alpha_s^2 \Lambda_{QCD} \left(\frac{1}{x}\right)^{\alpha_p - 1}$$

where we use $r_h \sim \frac{1}{\Lambda_{QCD}}$. The saturation momentum Q_s is a new dimensional scale in the problem. We conclude that a violation of unitarity occurs for $Q < Q_s$.

For very small Bjorken x (large Y), $Q_s \gg \Lambda_{QCD}$. This implies that the violation of unitarity starts at short distance of the order $1/Q_s$, which is still in the validity of perturbative QCD. Therefore, the unitarity problem has to be solved in the framework of perturbative QCD

The BFKL evolution creates partons of roughly the same transverse size. As we decrease x, the parton density grows. However are some critical x_{cr},

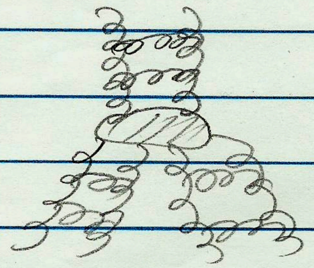
$$x_{cr} = \left(\frac{Q}{\alpha_s^2 \Lambda_{QCD}}\right)^{1/(\alpha_p - 1)}$$

corresponding to $Q_s(x_{cr}) = Q$, the density of partons in the transverse plane becomes so large and wave function of the partons start to overlap. For such a densely populated system, we need to take account interactions between the partons

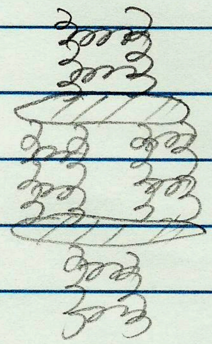
For $Q^2 < Q_s^2$, we should have a new evolution equation that includes interactions between the partons. This new evolution should slow down and finally stop (saturate) the increase in the number of "wee" partons, leading to the saturation of the parton density.

* The GLR-MQ equation

Consider a proton filled with various source of color charge (sea quarks and gluons), in such system, multiple BFKL ladder exchanges may become important. Ladder mergers should also possible



Since we are interested in the gluon distribution, which is a correlation function for two gluon fields, these multiple ladders should all merge in the end into a single ladder, leading to the "fan" diagrams.



pomeron loop. is argued to be subleading.

$2\to 1$ "fan" diagram would bring in a quadratic correction to the linear BFKL equation for the unintegrated gluon distribution leading to GLR evolution equation

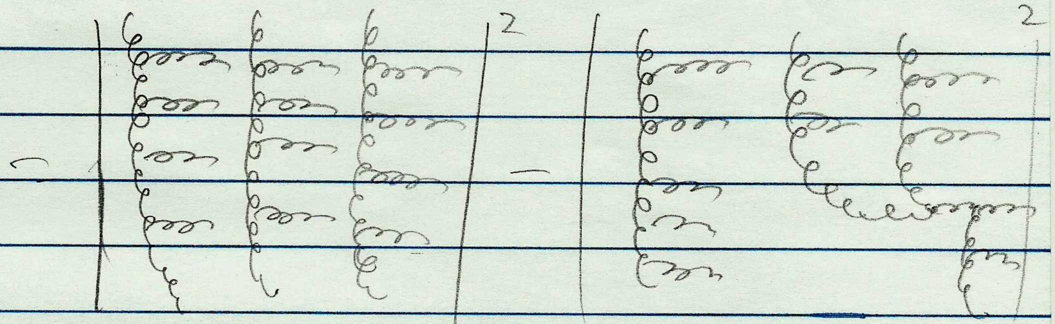
$$\frac{\partial \phi(x,k_\perp^2)}{\partial \ln(1/x)} = \frac{\alpha_s N_c}{\pi}\int d^2 l_\perp\, K_{BFKL}(k,l)\,\phi(x,k_\perp^2) - \frac{\alpha_s^2 N_c \pi}{2C_F S_\perp}\left[\phi(x,k_\perp^2)\right]^2$$

$S_\perp = \pi R^2$ is the cross sectional area of the proton or nucleus along beam direction. The quadratic term, responsible for ladder merges, introduces damping and slows down the growth of the gluon distributions with energy.

In double-leading-logarithmic approximation

$$xG(x,Q^2) = \int^{Q^2} dk_\perp^2\, \phi(x,k_\perp^2)$$

$$\frac{\partial^2 xG(x,Q^2)}{\partial \ln(1/x)\,\partial \ln(Q^2/\Lambda^2)} = \frac{\alpha_s N_c}{\pi} xG(x,Q^2) - \frac{\alpha_s^2 N_c \pi}{2C_F S_\perp}\frac{1}{Q^2}\left(xG(x,Q^2)\right)^2$$



Based on the non-linear equation, Q_s^2 can be more precisely estimated,

$$\frac{\alpha_s N_c}{\pi} xG(x,Q^2) = \frac{\alpha_s^2 N_c \pi}{2C_F S_\perp}\frac{1}{Q^2}\left(xG(x,Q^2)\right)^2.$$

$$Q_s^2 = \frac{\alpha_s \pi^2}{S_\perp 2C_F}\, xG(x,Q_s^2).$$

Since

$$xG \sim \left(\frac{1}{x}\right)^\lambda$$

For nucleus A,

$$Q_s^2 \sim \frac{A}{S_\perp}\left(\frac{1}{x}\right)^\lambda \sim A^{1/3}\left(\frac{1}{x}\right)^\lambda.$$

This $A^{1/3}$ is important: it means that the saturation scale is larger for large nucleus. We will observe the saturation region ($Q < Q_s$) is broader for DIS on a nucleus.

$$\frac{1}{Q_S^2} \rho_{glue}(x, Q_s^2) \sim \frac{1}{\alpha_S}. \qquad \rho_{glue} = \frac{xG}{S_\perp}$$

$$\Downarrow$$

$$A_\mu \sim \frac{1}{g}.$$

This means the saturation occurs because the gluon field gets as strong as it can possibly be, leading to the saturation of gluon field strength and parton distribution functions