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# High Energy QCD Journal Club

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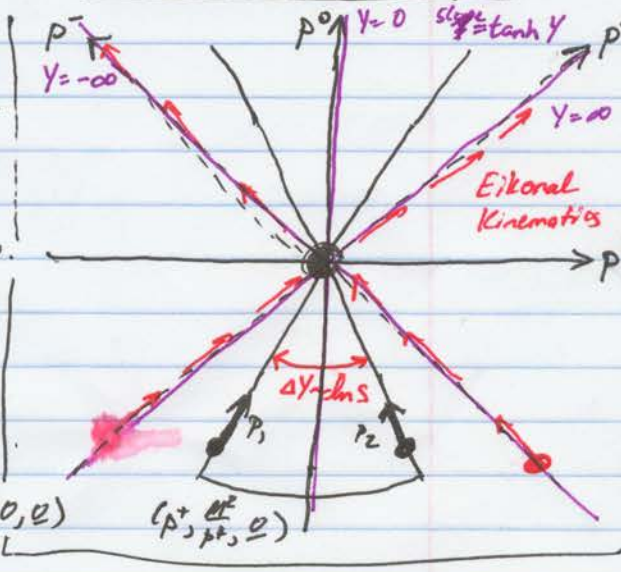
Lecture Notes: Small-x Evolution in DIS

This subject brings together 2 main themes:

- Multiple Rescattering (i.e., nucleus) → gluon saturation in IR.
- Formulation of quantum corrections as differential evolution eqns.

## I. The High Energy Limit of QCD




Consider two particles scattering with momenta $p_1$ and $p_2$ in CMS frame. The rapidity of a particle with momentum $p$ and mass $m$ is

$$y \equiv \frac{1}{2}\ln\frac{p^+}{p^-} = \ln\left(\frac{p^+}{\sqrt{p_T^2+m^2}}\right) = \ln\left(\frac{\sqrt{p_T^2+m^2}}{p^-}\right)$$

In the CMS frame $(p_{1T}^2 = p_{2T}^2 = 0)$, $(y_1 = -y_2 = \Delta y/2)$

$$\Delta y \equiv y_1 - y_2 = \ln\frac{p_1^+}{m} - \ln\frac{m}{p_2^-} = \ln\left(\frac{p_1^+ p_2^-}{m^2}\right) \approx \ln\frac{s}{\perp^2}$$

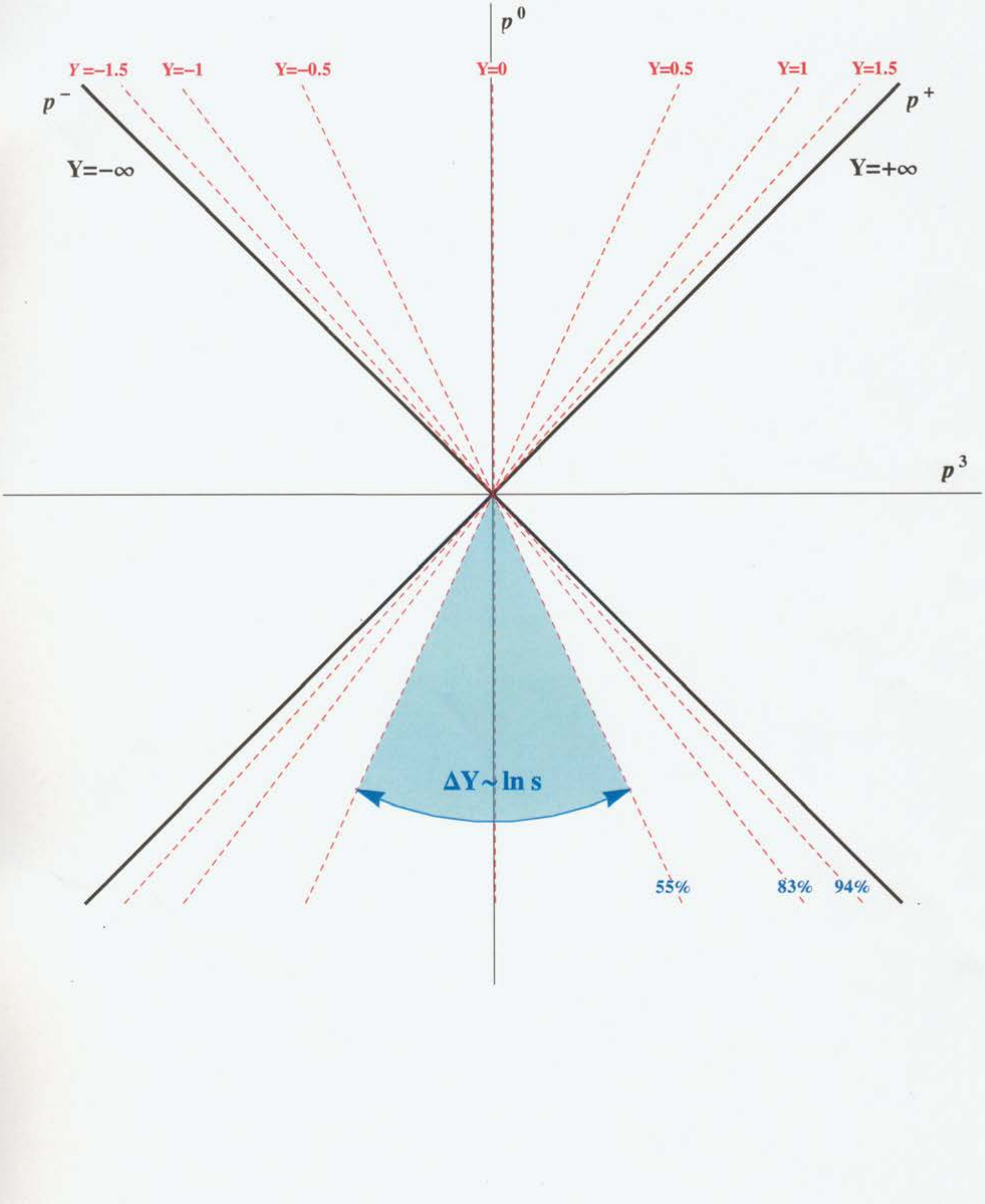
$$\boxed{\Delta y \sim \ln s}$$

As we increase $s$ or $\Delta y$, the kinematics rapidly converge:

$$\begin{cases} p_1^+ = m e^{+\frac{1}{2}\Delta y} & p_1^- = m e^{-\frac{1}{2}\Delta y} \\ p_2^+ = m e^{-\frac{1}{2}\Delta y} & p_2^- = m e^{+\frac{1}{2}\Delta y} \end{cases} \qquad \underline{s \sim p_1^+ p_2^- \sim m^2 e^{\Delta y}}$$

Thus, the $\Delta y \to \infty$ high energy kinematics correspond to the hierarchy of scales

$$s = p_1^+ p_2^- \gg \perp^2 \gg \perp^4/s$$



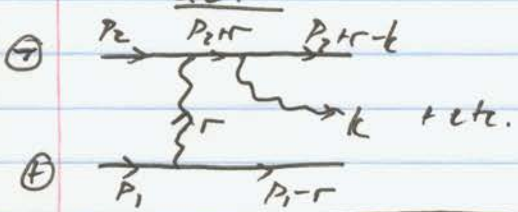
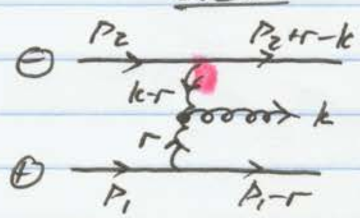
$p^0$
Y=-1.5
Y=-1
Y=-0.5
Y=0
Y=0.5
Y=1
Y=1.5
$p^-$
$p^+$
Y=-∞
Y=+∞
$p^3$
ΔY ~ ln s
55%
83%
94%

The momenta quickly converge to

$$p_1^\mu = (p^+, 0, \underline{0})$$
$$p_2^\mu = (0, p^-, \underline{0})$$

after only 1 or 2 units of rapidity. These <u>frozen high-energy kinematics</u> are referred to as <u>"eikonal"</u> kinematics, in which the two particles travel in straight line trajectories throughout the interaction.

Why is the high-energy limit of QCD interesting?
- consider the high-energy scattering of 2 charges in QED vs. QCD. Is any radiation emitted?

| <u>QED</u> | <u>QCD</u> |
|---|---|
|  |  |
| • $\sum$(diagrams) = 0<br>• No EM field emitted<br>• No recoil of particles (no acceleration)<br>• Constant lines of charge | • $\sum$(diagrams) $\neq 0$<br>• Important non-Abelian dynamics give radiation, without recoil.<br>• Acceleration in color space |

The <u>high-energy</u> limit is sensitive to <u>QCD color dynamics</u>!

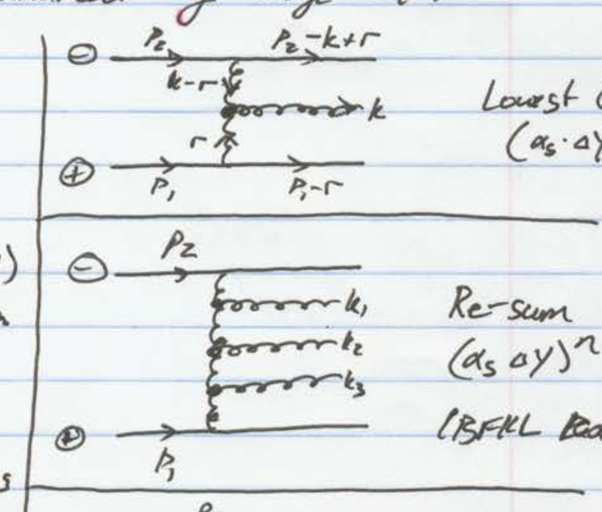
We take the high-energy eikonal kinematics by using $s \gg \perp^2$, which sets in after only 1 or 2 units of rapidity. A calculation in this regime is described by the tree-level diagram with $\mathcal{O}(\alpha_s)$ perturbative corrections.

But there is a characteristic scale of the quantum corrections: $\alpha_s \ll 1$. But at high energies, $\Delta Y \sim \ln s$ becomes a compensating factor that offsets the power of $\alpha_s$.

- Quantum corrections that are systematically enhanced by $\Delta Y$: $(\alpha_s \cdot \Delta Y)$ vs $(\alpha_s)$ re-order the perturbation series.

- At $\Delta Y \gtrsim \frac{1}{\alpha_s}$, or $s \gtrsim \perp^2 \cdot e^{1/\alpha_s} \gg \perp^2$, these enhanced quantum corrections become $\mathcal{O}(1)$ and re-shape the ground state / lowest order interactions.
  - Moving from one effective theory to another occurs by perturbatively restructuring the perturbation series.

- What quantum corrections are enhanced by large $\Delta Y$?




Extra emission of soft gluons:
$k^+ \ll p_1^+$ or $\boxed{x \ll 1}$

$$\sigma_{qq \to qqG} \sim \int \frac{\alpha_s^3 \, d^2k \, d^2r}{k_T^2 \, r_T^2 \, (\underline{k} - \underline{r})^2} \cdot \int_0^{\Delta Y} dy \sim \underset{\text{correction}}{\mathcal{O}(\alpha_s \Delta Y)}$$

- Small-$x$ / longitudinally soft gluons
  - do not disturb eikonal kinematics
  - independent of $y$ once soft
  - enhanced due to large "soft gluon phase space"

- Successive gluon emissions are all enhanced if they are successively softer:

$p_1^+ \gg k_1^+ \gg k_2^+ \gg k_3^+ \ldots$ (longitudinal ordering)

- Calculate one step of small-x evolution: emit one soft gluon.
- Write a <u>differential</u> <u>eqn.</u> to relate the amplitude to itself recursively after gluon emission.
- The <u>solution</u> to this eqn. re-sums the <u>leading logarithmic enhancements</u> (LLA).

<u>Physical Pictures</u>:

- At very high energies, relativistic color charges undergo a <u>cascade</u> of <u>longitudinally soft gluons</u>.
- Need to solve for the behavior of an n-gluon Fock state.

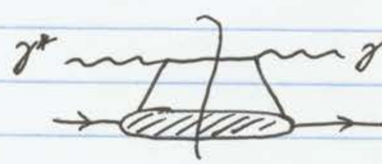
## II. Deep Inelastic Scattering at Increasing Energies

DIS is $e^- + A \to e^- + X$, or $\gamma^* + A \to X$ scattering, where the photon virtuality $Q^2 \gg \Lambda_{QCD}^2$ is large.

1) <u>Low Energy Scattering</u>

$s \sim \perp^2 \gg \Lambda_{QCD}^2$

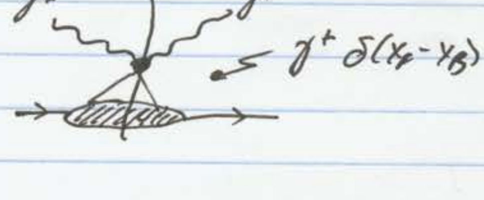
- Nonlocal E&M scattering




2) <u>Eikonal Scattering</u>

$s \gg \perp^2 \gg \Lambda_{QCD}^2$

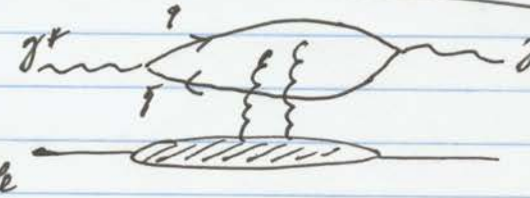
- QED vertex becomes local
- Measures PDF




3) <u>QCD Fluctuations</u>

$s \gg \perp^2/\alpha_s^2 \gg \perp^2 \gg \Lambda_{QCD}^2$

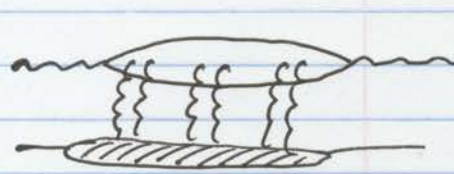
- $\gamma^*$ fluctuates into long-lived $q\bar{q}$ dipole
- Scattering occurs in QCD by t-channel gluons



4.) <u>QCD Fluctuations & Rescattering (Nucleus)</u>

$S \gg \perp^2/\alpha_s^2 \gg \perp^2 \gg \Lambda_{QCD}^2$

& $S \gg \perp^2 \cdot A^{1/3} \gg \perp^2 \gg \Lambda_{QCD}^2$

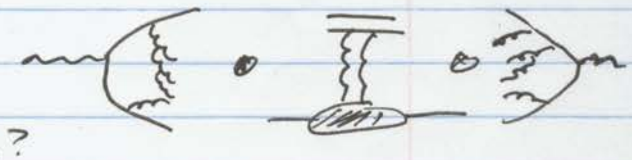


(Glauber-Mueller)

- Rescattering is enhanced by combinatorics.
- $\alpha_s^2 A^{1/3} \sim 1$
- Saturation screens out dipole S-matrix in the IR

5.) <u>QCD + Quantum Corrections</u>

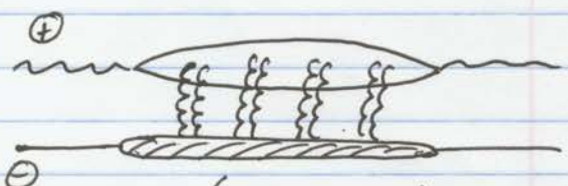
$S \gg \perp^2 e^{1/\alpha_s} \gg (\perp^2/\alpha_s^2 \sim \perp^2 \cdot A^{1/3}) \gg \perp^2 \gg \Lambda_{QCD}^2$



- Quantum evolution generates gluon cascade?
- Multiple rescattering and saturation?
  - ↳ How does quantum evolution interplay with saturation?

This last case is our objective: to <u>understand how</u> <u>quasiclassical saturation</u> effects are <u>dressed by LLA</u> <u>quantum corrections</u>.

## <u>III. The Gluon Cascade</u>



($A^+ = 0$ gauge)

Starting with Glauber-Mueller rescattering, look for quantum corrections that are enhanced by $\ln \frac{1}{x} \sim \ln s \sim \Delta Y$.

<u>Corrections not enhanced by $\Delta Y$:</u>

- Extra t-channel particle exchange
- Emission of s-channel quarks (beyond LLA)
- Gluon emission during / between interactions.

<u>Corrections that are enhanced by $\Delta Y$:</u>

- Emission of longitudinally soft gluons from the projectile (real & virtual)

Within the Glauber-Mueller model, the rescattering of many particles is <u>independent</u> to leading order in $A^{1/3}$.

<u>Physical Picture</u>: Projectile dipole undergoes small-x gluon cascade into many n-gluon Fock states. These particles all rescatter independently with the Glauber-Mueller amplitudes.

↳ Calculate small-x evolution of projectile wave fn.

<u>Christian's Derivation</u>

- Calculate $q\bar{q}G$ wave fn from LCPT (eq. 4.56)
  - ↳ derive (4.60), (4.61), (4.63)
- $\Psi(q\bar{q}G)$ proportional to $\Psi(q\bar{q})$
- Factor = conditional probability of gluon emission
- Logarithmic integral $\frac{dz}{z} \sim \ln\frac{1}{x} \sim \Delta Y$
- Real & virtual (4.64) corrections
- How to iterate? Successive emissions give a <u>hierarchy</u> of $n \to (n+1)$ gluon Fock states.

## <u>IV. The Large-$N_c$ Limit: Dipole Cascades</u>

QCD is an $N_c = 3$ representation of the general $SU(N_c)$ algebra. The <u>t'Hooft large-$N_c$ limit</u> is a formal limit of the algebra in which $N_c$ is a large parameter:

$$\alpha_s \ll 1 \quad ; \quad N_c \gg 1 \quad ; \quad \alpha_s N_c \sim 1$$

This limit reorders the perturbation series so that contributions that are $1/N_c$ suppressed are equivalent to $\mathcal{O}(\alpha_s)$ quantum corrections.

(#) See Lance Dixon Notes

The large-$N_c$ limit is also related to a supersymmetric $N=4$ Super Yang-Mills theory at tree level. (#) This theory has been used to study interactions at strong coupling. To understand the large-$N_c$ limit, consider the relationship between the $U(N_c)$ and $SU(N_c)$ algebras.

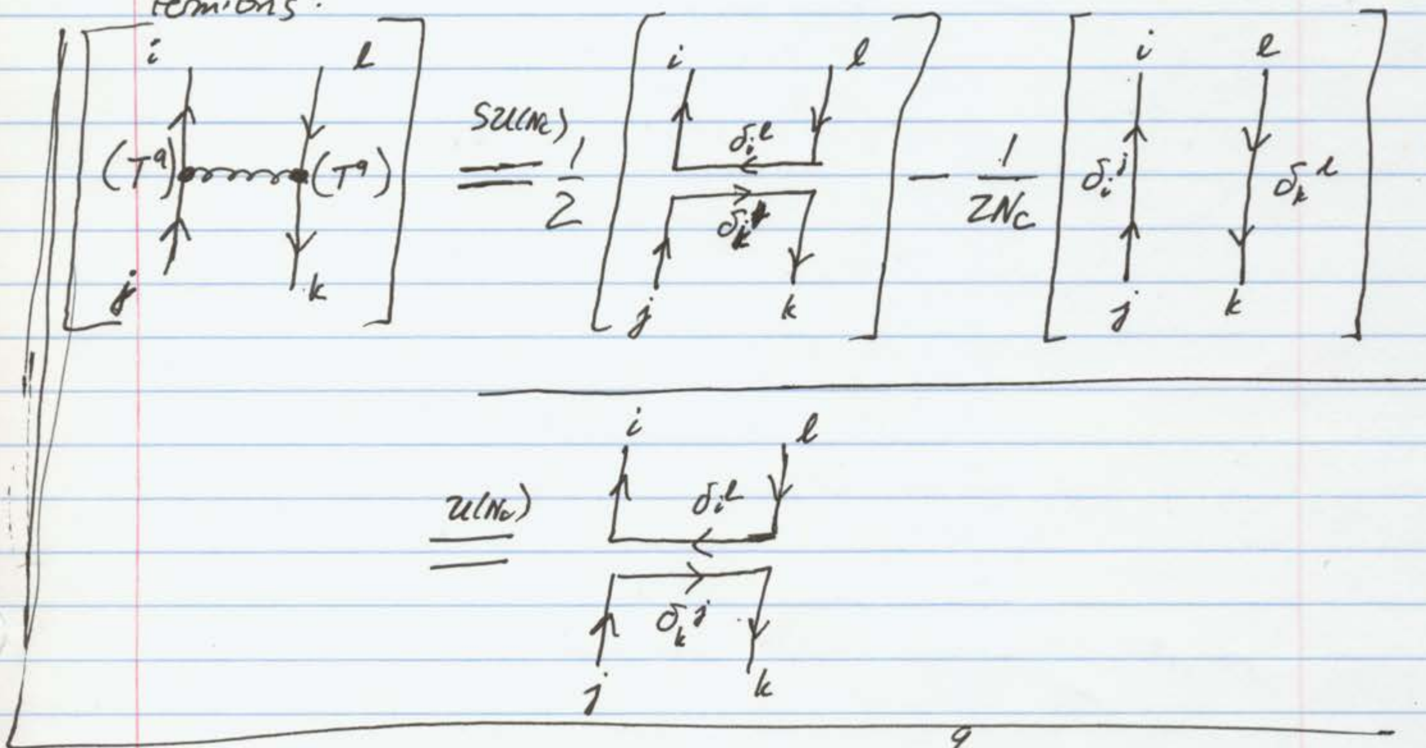
| $U(N_c)$ | $SU(N_c)$ |
| --- | --- |
| • Unitary $N_c \times N_c$ matrices | • Unitary $N_c \times N_c$ matrices |
| • $N_c^2$ degrees of freedom | • (Det = 1) constraint |
| • $N_c^2$ degrees of freedom | • $N_c^2 - 1$ degrees of freedom |
| ↳ $N_c^2$ Generators [Adjoint Repsn.] | ↳ $(N_c^2-1)$ Generators [Adjoint Repsn.] |
| ↳ All generators (Hermitian) $T^a{}_{(N_c \times N_c)}$ | ↳ Traceless generators (Hermitian) $T^a{}_{(N_c \times N_c)}$ |
| • Includes $\mathbb{1}$ | • Excludes $\mathbb{1}$ |
| • Fierz identity: $(T^a)_i{}^j (T^a)_k{}^l = \frac{1}{2}\delta_i{}^l \delta_k{}^j$ | • Fierz identity: $(T^a)_i{}^j (T^a)_k{}^l = \frac{1}{2}\delta_i{}^l \delta_k{}^j - \frac{1}{2N_c}\delta_i{}^j \delta_k{}^l$ |

- The difference between $U(N_c)$ and $SU(N_c)$ is the addition of a new generator proportional to $\mathbb{1}$.
  - ↳ This essentially adds an Abelian $U(1)$ extra gauge particle
    - ↳ It adds a "photon" with coupling $g$ to complement the non-Abelian gluons.

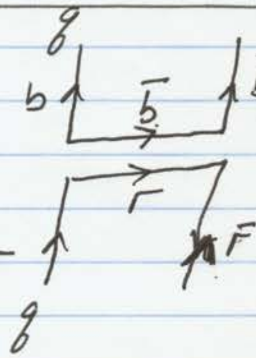
- The color flow of these two groups is encoded in their Fierz identities:

$$U(N_c): \quad (T^a)_i{}^j (T^a)_k{}^l = \frac{1}{2}\delta_i{}^l \delta_k{}^j$$
$$SU(N_c): \quad (T^a)_i{}^j (T^a)_k{}^l = \frac{1}{2}\delta_i{}^l \delta_k{}^j - \frac{1}{2N_c}\delta_i{}^j \delta_k{}^l$$

Interpreted graphically, these identities tell us how the gluons/photon carry color between the fermions:

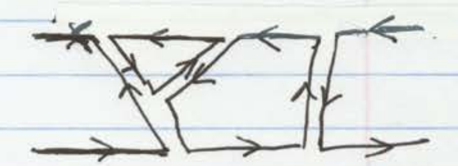



- In the large-$N_c$ limit, we essentially complete the $SU(N_c)$ group to $U(N_c)$. This fits the naive conception of $(r\bar{b}, rg, \text{etc.})$ dual color gluons.




- Gluons are represented as two quarks in an octet state.

$\Rightarrow$ Unifies quarks + gluons into color dipoles. $\rightarrow$



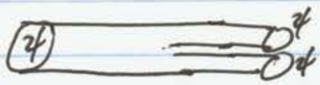
Any superposition of quarks & gluons can be represented by a set of independent color dipoles.

## Features of the large-$N_c$ limit

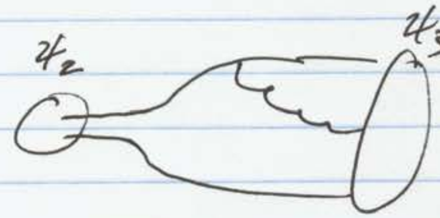
- Unification of gluons → dipoles. (Eliminates hierarchy)
- Non-planar diagrams are suppressed by $1/N_c$.

eg)

- Re-Express one step of small-$x$ evolution as dipole splitting:



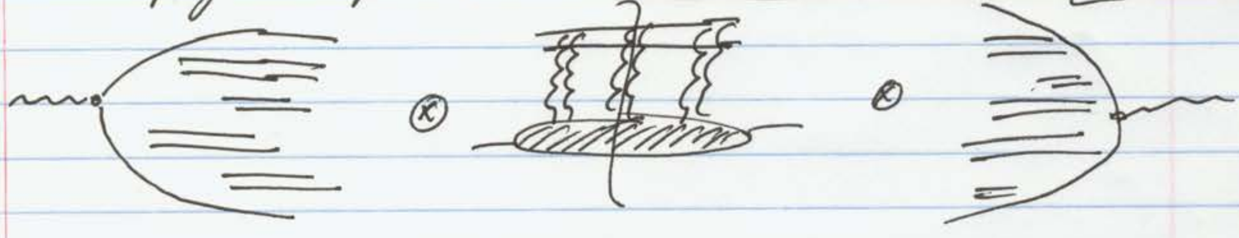
vs



$$\boxed{\text{One step of evolution = one dipole emission (real or virtual corrections).}}$$

## V. The Balitsky-Kovchegov Equation

- The physical picture: $\left(\gamma^* + A \to X\right)^2 \sim \text{Im}\left[\gamma^* + A \to \gamma^* + A\right]$



- Work in rest frame of target, $A^+ = 0$ gauge → all evolution in projectile wavefn.
- Express wavefn as a distribution of dipoles to leading order in $N_c$.
- Dipoles scatter independently on the target with Glauber-Mueller amplitudes

$$S^{(0)}_{dip}(\underline{r}, \underline{b}) = e^{-\frac{1}{4} r_\perp^2 Q_{s0}^2(\underline{b}) \left[\ln \frac{1}{r_\perp \Lambda}\right]}$$

· Dipole Wave Function: (Mueller Dipole Model)

Write a "partition fn" or "dipole generating functional" to describe the weights assigned to the $n$-dipole Fock states:

$$Z(\underline{r}_0, \underline{b}_0, Y; [u]) \equiv \frac{\sum_n \frac{1}{n!} \int d\Gamma_n \, |\psi^{[n]}|^2 \cdot u_1 \cdots u_n}{|\psi_{bare}|^2}$$

which describes all the features of the dipole wave function in terms of its total separation vector, impact parameter, and rapidity.

The functions $\{u_n\}$ are dummy basis functions for differentiation.

· Properties of $Z$:

1) $$\left. \frac{\delta Z(r_0, b_0)}{\delta u(\underline{r}, \underline{b})} \right|_{u=1} = \langle n_1 \rangle (r_0, b_0, r, b, Y)$$

average # of dipoles with given parameters in wavefn.

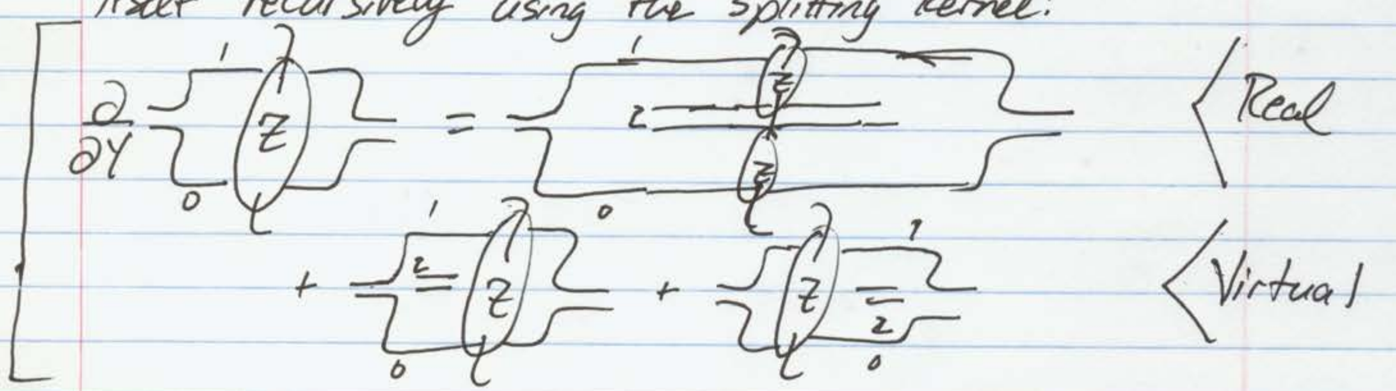
2) $$\left. \frac{\delta^n Z}{\delta u_1 \cdots \delta u_n} \right|_{u=0} = \frac{|\psi^{[n]}|^2}{|\psi_{bare}|^2}$$

projects out $n$-dipole Fock state.

3) $$\left. \frac{\delta^n Z}{\delta u_1 \cdots \delta u_n} \right|_{u=1} = \langle N_n \rangle$$

average # of pairs/triplets/$n$-tuplets of dipoles [correlation function].

To resum the small-$x$ evolution, relate $Z$ to itself recursively using the splitting kernel:




$r_0 = x_1 - x_0$

$$\frac{\partial}{\partial Y} Z(\underline{r_0}, \underline{b_0}, Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{(x_0 - x_1)^2}{(\underline{x_0 - x_2})^2 (\underline{x_2 - x_1})^2} \Big[ Z(\underline{x_1 - x_2}, \underline{b}, Y) \, Z(\underline{x_0 - x_2}, \underline{b}, Y) - Z(\underline{x_0 - x_1}, \underline{b}, Y) \Big]$$

Mueller Dipole Model

Using this, convolute with Glauber-Mueller dipole scattering amplitudes:

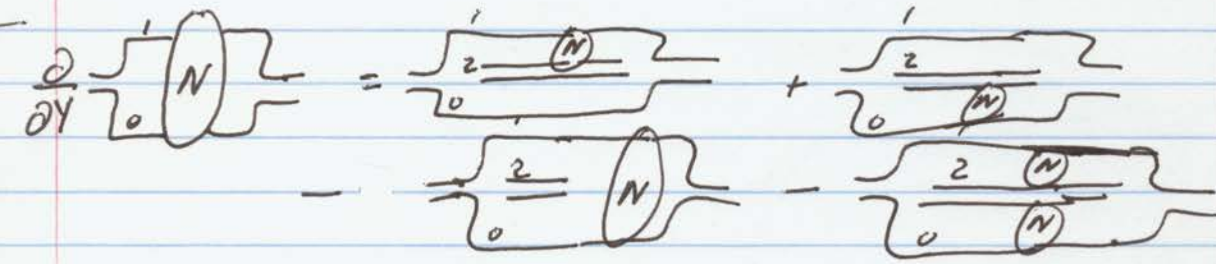
$$S_{tot}(\underline{r_0}, \underline{b_0}, Y) = \Sigma_n \frac{1}{n!} \int d\Gamma_n \, |\Psi^{[n]}|^2 \cdot s_0(\underline{r_1}, \underline{b_1}) \cdots s_n(\underline{r_n}, \underline{b_n})$$
$$= \Sigma_n \frac{1}{n!} |\Psi_{q\bar{q}}|^2 \left( \frac{\delta^n Z}{\delta u_1 \cdots \delta u_n} \right) s_0(\underline{r_1}, \underline{b_1}) \cdots s_n(\underline{r_n}, \underline{b_n})$$

$\Rightarrow$ $S_{tot}$ plays the role of $Z$, using the dipole scattering amplitudes as basis fns.

$\hookrightarrow$ $S$ obeys the same differential eqn. as $Z$.

Rewriting this in terms of $N$, the imaginary part of the forward dipole scattering amplitude

$N = 1 - S$ gives




$$\frac{\partial}{\partial Y} N(x_{10}, \underline{b}, Y) = \frac{\alpha_s N_c}{2\pi^2}\int d^2x_2 \, \frac{x_{10}^2}{x_{20}^2 x_{21}^2}\Big\{ N(\underline{x}_{12}, \underline{b}, Y) + N(\underline{x}_{20}, \underline{b}, Y) - N(\underline{x}_{12}, \underline{b}, Y) - N(\underline{x}_{12}, \underline{b}, Y) \cdot N(\underline{x}_{20}, \underline{b}, Y) \Big\}$$

Balitsky – Kovchegov Eqn. (BK)

- The BK eqn. resums the dipole - nucleus interaction with the nucleus in the LLA.

If the bare amplitude is

$$n_0 = 1 - e^{-\frac{1}{4} r_T^2 Q_{s0}^2 \ln \frac{1}{r_T \Lambda}}$$

then the full, $Y$-dependent solution to the BK eqn. defines

$$N(Y) \equiv 1 - e^{-\frac{1}{4} r_T^2 Q_s^2(Y) \ln \frac{1}{r_T \Lambda}}$$

where $Q_s^2(Y)$ is the LLA solution for rapidity-dependent quantum corrections to the Glauber – Mueller picture of saturation.

## VI. Solution to the BK - Equation