High-Energy QCD Journal Club

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Lecture Notes: $\boxed{\text{Small-}x \text{ Evolution in DIS}}$ (Part II)

Previously:
- Reviewed the high-energy limit: eikonal kinematics + quantum evolution
- Formulated small-x evolution in DIS as a dipole cascade of γ^* wave function in the large-N_c limit.
- Expressed dipole wave-fn. in the Mueller dipole model by a generating functional Z
- Wrote down evolution eqn. in Z, S, and N for small-x evolution.

BK equation:

$$\left[\; \frac{\partial}{\partial Y} N(\underline{x}_{01}, \underline{b}, Y) = \frac{\alpha_s C_F}{\pi^2} \int d^2x_2 \, \frac{x_{01}^2}{x_{02}^2 x_{21}^2} \Big[N(\underline{x}_{02}, \underline{b} + \tfrac{1}{2}\underline{x}_{21}, Y) + N(\underline{x}_{12}, \underline{b} + \tfrac{1}{2}\underline{x}_{20}, Y) - N(\underline{x}_{01}, \underline{b}, Y) - N(\underline{x}_{02}, \underline{b} + \tfrac{1}{2}\underline{x}_{21}, Y) \cdot N(\underline{x}_{12}, \underline{b} + \tfrac{1}{2}\underline{x}_{20}, Y) \Big] \right.$$

- Neglecting the small differences in impact parameter, this is

$$\left[\; \frac{\partial}{\partial Y} N(\underline{x}_{01}, Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \, \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \Big[N(\underline{x}_{02}, Y) + N(\underline{x}_{12}, Y) - N(\underline{x}_{01}, Y) - N(\underline{x}_{12}, Y) \cdot N(\underline{x}_{02}, Y) \Big] \right.$$

There are 2 fixed points of the BK eqn:

• $N = 0 = \text{const}$	Transparent limit	Unstable
• $N = 1 = \text{const}$	Black disk limit	Stable

· To get a feel for BK, consider the atrocious approximation of neglecting all x_T dependence:

$$\frac{dN(Y)}{dY} = \omega(N - N^2) \qquad \text{for} \quad \omega > 0$$

· This reduces BK to a (separable) logistic eqn. The solution is:

$$\frac{dN}{N-N^2} = \frac{-dN}{N(N-1)} = -dN\left(\frac{-1}{N} + \frac{1}{N-1}\right) = \omega\, dY$$

$$\hookrightarrow \quad \int_{N_0}^{N} dN \left(\frac{1}{N} - \frac{1}{N-1}\right) = \int_0^Y \omega\, dY$$

$$\hookrightarrow \quad \left[\ln \frac{N}{N-1}\right]_{N_0}^{N} = \omega Y$$

$$\hookrightarrow \quad \ln\left(\frac{N}{N-1} \cdot \frac{N_0 - 1}{N_0}\right) = \omega Y$$

$$\hookrightarrow \quad \frac{N}{N-1} = \left(\frac{N_0}{N_0 - 1}\right) e^{\omega Y}$$

$$\hookrightarrow \quad N\left[1 - \frac{N_0}{N_0 - 1} e^{\omega Y}\right] = -\left(\frac{N_0}{N_0 - 1}\right) e^{\omega Y}$$

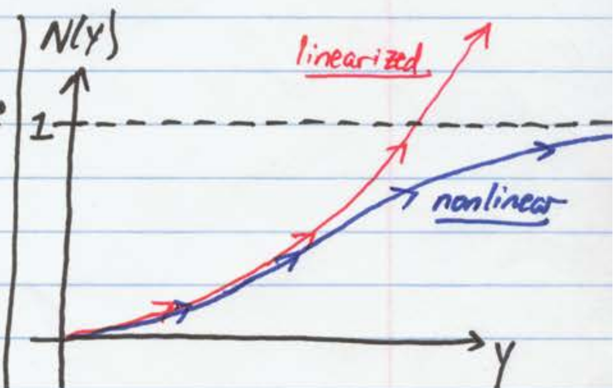
$$\hookrightarrow \quad \boxed{N(Y) = \frac{-N_0\, e^{\omega Y}}{N_0 - 1 - N_0 e^{\omega Y}} = \frac{N_0\, e^{\omega Y}}{1 + N_0\left(e^{\omega Y} - 1\right)}}$$

· If, instead, we had linearized the eqn. by dropping N^2, we would have

$$\frac{dN}{dY} = \omega N$$

with exponential solution

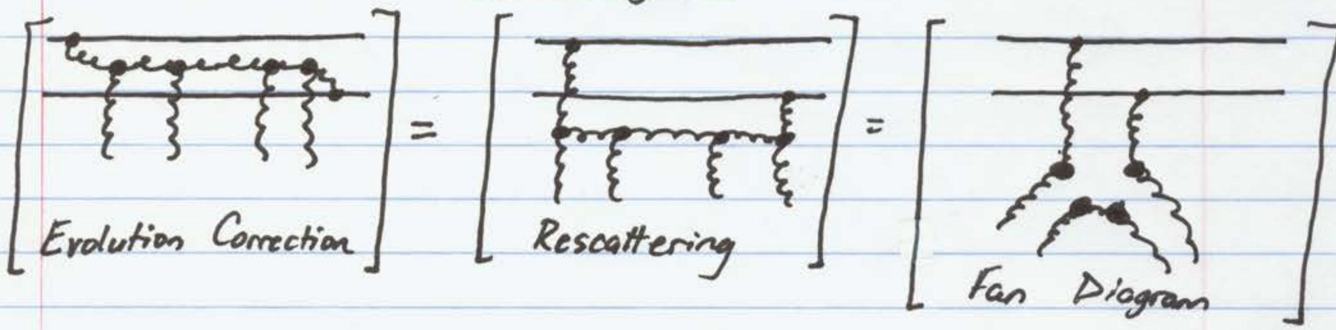
$$\underline{N(Y) = N_0\, e^{\omega Y}}$$



$$\lim_{Y \to \infty} N(Y) = 1$$

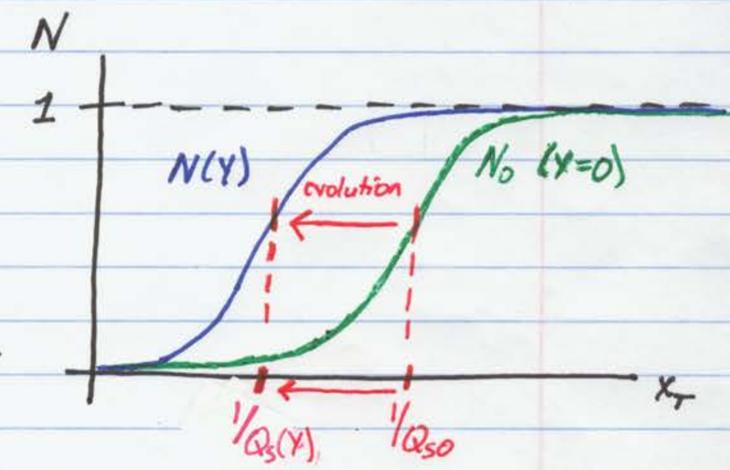
- Even in this oversimplified form, we see that the nonlinear term in the evolution eqn. cuts off the growth of N with Y (or s) and restores unitarity.

- The nonlinear term corresponds to multiple rescattering in the form of fan diagrams which are resummed:



- The initial condition for BK is given by the quasi-classical Glauber-Mueller expression

$$\boxed{N_0(x_T) = 1 - \exp\left[-\tfrac{1}{4} x_T^2 Q_S^2 \ln \tfrac{1}{x_T \Lambda}\right]}$$



- The solution to BK will define a rapidity-dependent saturation scale $Q_S(Y)$:

$$\boxed{N(x_T, Y) \equiv 1 - \exp\left[-\tfrac{1}{4} x_T^2 Q_S^2(Y) \ln \tfrac{1}{x_T \Lambda}\right]}$$

- BK belongs to a universality class of equations. Many approximations will capture the correct onset of saturation, only changing the details of the "knee".

- There are <u>no known exact solutions</u> to BK, but we can solve for the <u>asymptotic behavior</u> near $N=0$ and $N=1$.

<u>VII. The Dilute BFKL Limit</u>

- For small dipoles $x_T \to 0$, the scattering amplitude is small, $N \ll 1$, so we can <u>linearize</u> the BK eqn. back to the <u>BFKL</u> eqn.

Drop the nonlinear N^2 term:

$$\frac{\partial}{\partial Y} N(\underline{x}_{01}, Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \, \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[N(\underline{x}_{02}, Y) + N(\underline{x}_{12}, Y) - N(\underline{x}_{01}, Y) \right]$$

- As a further simplification, consider <u>integrating out the angular dependence</u>, so that N depends only on the dipole sizes:

$$\frac{\partial}{\partial Y} N(x_{01}, Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \, \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[N(x_{12}, Y) + N(x_{20}, Y) - N(x_{01}, Y) \right]$$

- The BFKL kernel (which is the same as in BK) possesses <u>conformal symmetry</u>. This strongly constrains the form of the eigenfunctions.

  - Map the transverse plane to the <u>complex plane</u>:
    $\rho \equiv x + iy$; $\rho^* \equiv x - iy$
  - The kernel is invariant under the <u>conformal Möbius group</u>:

    $$\rho \to \frac{a\rho + b}{c\rho + d} \quad \text{for all } a,b,c,d \in \mathbb{C}$$

    which includes <u>rotations</u>, <u>translations</u>, <u>reflections</u>, and <u>scale dilations</u>.

• We can use the scaling properties to show that <u>powers of the dipole size</u> are eigenfunctions of the kernel:

$$N_\lambda(x_T, Y) = x_T^\lambda \cdot f_\lambda(Y) \quad \text{for some } \lambda$$

Then

$$x_{10}^\lambda f_\lambda'(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \, \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[x_{12}^\lambda + x_{20}^\lambda - x_{10}^\lambda \right] f_\lambda(Y)$$

$$= \frac{\alpha_s N_c}{2\pi^2} \underbrace{(x_{10}^\lambda)}_{\substack{x_{10}^\lambda \text{ is an} \\ \text{eigenfunction}}} f_\lambda(Y) \cdot \underbrace{\int d^2x \, \frac{x_{10}^2}{x_{12}^2 x_{20}^2} \left[\left(\frac{x_{12}}{x_{10}}\right)^\lambda + \left(\frac{x_{20}}{x_{10}}\right)^\lambda - 1 \right]}_{\substack{\text{This integral is } x_{10}\text{-independent} \\ \text{and determines the eigenvalue:} \\ 2\pi \cdot \chi(\lambda) \text{ for } \lambda = 1+2i\nu}}$$

$$x_{10}^{1+2i\nu} f_\nu'(Y) = \frac{\alpha_s N_c}{2\pi^2} \, x_{10}^{1+2i\nu} f_\lambda(Y) \cdot 2\pi \, \chi(0,\nu) \qquad \underline{\nu = \text{real}}$$

where the eigenvalue is

(See plot)

$$\boxed{\chi(0,\nu) = 2\psi(1) - \psi\left(\tfrac{1}{2} + i\nu\right) - \psi\left(\tfrac{1}{2} - i\nu\right)} \qquad (\underline{\text{Real}})$$

where $\psi(z) \equiv \frac{d}{dz} \ln[\Gamma(z)]$ is the <u>digamma function</u>.

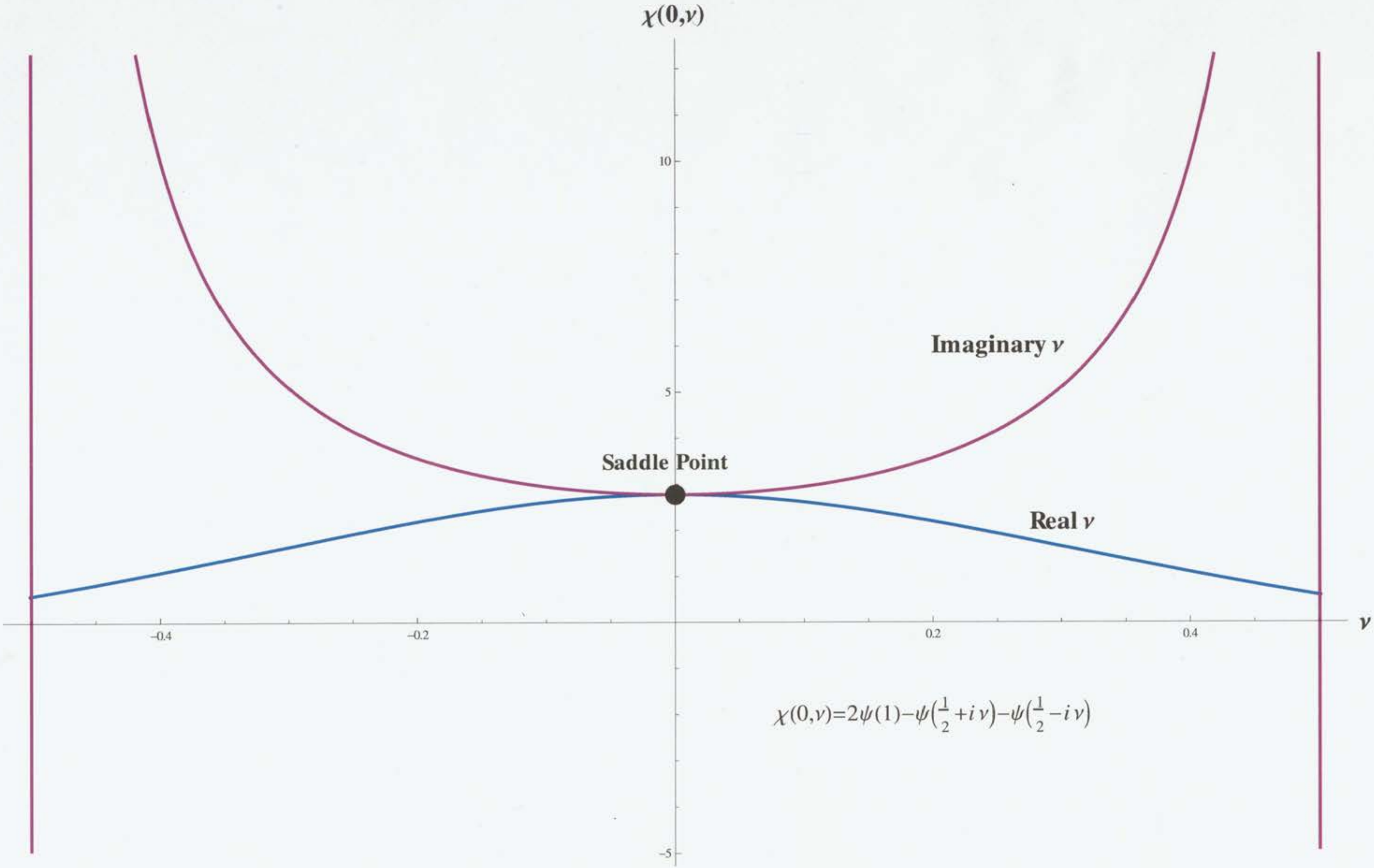
This gives

$$f_\nu'(Y) = \frac{\alpha_s N_c}{\pi} \chi(0,\nu) \, f_\nu(Y)$$

$$\hookrightarrow \boxed{f_\nu(Y) = C_\nu \, e^{\bar{\alpha}_s \chi(0,\nu) Y}} \qquad \left(\bar{\alpha}_s \equiv \alpha_s N_c / \pi \right)$$

so the general solution of the BFKL eqn is

$$\boxed{\begin{aligned} N(x_T, Y) &= \int d\nu \, C_\nu \, x_T^{1+2i\nu} \, e^{\bar{\alpha}_s \chi(0,\nu) Y} \\ &= \int d\nu \, \tilde{C}_\nu \, \exp\left[\bar{\alpha}_s \chi(0,\nu) Y + (1+2i\nu) \ln(x_T Q_{s0})\right] \end{aligned}}$$



χ(0,ν)
10
5
-5
Imaginary ν
Saddle Point
Real ν
-0.4
-0.2
0.2
0.4
ν
$\chi(0,\nu)=2\psi(1)-\psi\left(\frac{1}{2}+i\nu\right)-\psi\left(\frac{1}{2}-i\nu\right)$

where the coefficients C_γ are fixed by the initial condition

$$N(x_T, Y=0) = 1 - \exp\left[-\tfrac{1}{4} x_T^2 Q_{s0}^2\right]$$

See plot of χ

- We can approximate the behavior at large Y by using a <u>saddle point expansion</u>.

- The saddle point condition is

$$0 = \frac{d}{d\gamma}\left[\text{exponent}\right] = \boxed{\bar{\alpha}_s \chi'(0,\gamma_{sp}) Y + 2i \ln(x_T Q_{s0}) = 0}$$

$$\boxed{\gamma_{sp}^* \approx \frac{i \ln(x_T Q_{s0})}{14 \zeta(3) \bar{\alpha}_s Y}}$$

which fixes the saddle point γ_{sp} near 0. Use the zero$^{\text{th}}$ order approximation of the $d\gamma$-integral ($\gamma = \gamma_{sp}$) to estimate

$$N(x_T, Y) \sim \exp\left[\bar{\alpha}_s \underbrace{\chi(0,\gamma_{sp})}_{\approx 4\ln 2} Y + (1+2i\gamma_{sp})\ln(x_T Q_{s0})\right]$$

- Let's use these $N \to 0$ asymptotics to <u>estimate the transition to $N \sim O(1)$</u> when the linear BFKL evolution breaks down:

$$N \sim O(1) \text{ when } \quad \boxed{\bar{\alpha}_s \chi(0,\gamma_{sp}) Y + (1+2i\gamma_{sp}) \ln(x_T Q_{s0}) \approx 0}$$

- Use this to <u>define the rapidity-dependent saturation scale</u> $x_T = 1/Q_s(Y)$:

$$\begin{cases} (1) & \bar{\alpha}_s \chi'(0,\gamma_0) Y + 2i \ln Q_{s0}/Q_s(Y) = 0 \\ (2) & \bar{\alpha}_s \chi(0,\gamma_0) Y + (1+2i\gamma_0) \ln Q_{s0}/Q_s(Y) = 0 \end{cases}$$

where $\gamma_0 = \gamma_{sp}$ at $x_T = 1/Q_s(Y)$

- Dividing (1) and (2) gives

$$\frac{\chi'(0,\gamma_0)}{\chi(0,\gamma_0)} = \frac{2i}{1+2i\gamma_0}$$

and solving (1) for $Q_s(Y)$ gives

$$\boxed{\begin{aligned} Q_s(Y) &= Q_{s0} \exp\left[\bar{\alpha}_s Y \frac{\chi'(0,\gamma_0)}{2i}\right] \\ &= Q_{s0} \exp\left[\bar{\alpha}_s Y \frac{\chi(0,\gamma_0)}{1+2i\gamma_0}\right] \end{aligned}}$$

for $\gamma_0 \approx 0$, $\chi(0,0) = 4\ln 2$

$\underline{Q_s(Y) \approx Q_{s0}\, e^{2.77\bar{\alpha}_s Y}}$

This gives an estimate of the saturation scale that grows as e^Y or, equivalently, a power of s.

- Additionally, if we plug this back into the forward amplitude by writing

$$e^{\bar{\alpha}_s Y} = \left[Q_s(Y)/Q_{s0}\right]^{\frac{1+2i\gamma_0}{\chi(0,\gamma_0)}}$$

$$\hookrightarrow N(x_T, Y) \sim (x_T Q_{s0})^{1+2i\gamma_{sp}} \left[e^{\bar{\alpha}_s Y}\right]^{\chi(0,\gamma_{sp})}$$

$$\sim (x_T Q_{s0})^{1+2i\gamma_{sp}} \left[\frac{Q_s(Y)}{Q_{s0}}\right]^{(1+2i\gamma_0)\frac{\chi(0,\gamma_{sp})}{\chi(0,\gamma_0)}}$$

If $x_T \ll 1/Q_s(Y)$, then $\gamma_0 \neq \gamma_{sp}$. But if $x_T \lesssim 1/Q_s(Y)$, then $\gamma_0 \approx \gamma_{sp}$, and

$$\boxed{N(x_T, Y) \sim \left[x_T Q_s(Y)\right]^{1+2i\gamma_0}}$$

The amplitude is a function of a single <u>scaling variable</u>
$\tau = x_T Q_s(Y)$ or $\tau^{-1} \sim k_T/Q_s(Y)$
which is a signature that <u>the physics is dominated by a single scale</u> $Q_s(Y)$. Remarkably, this is still true even outside of the saturation scale.

- This phenomenon is valid outside of the saturation regime and is known as <u>extended geometric scaling</u>.

- We can estimate where this breaks down by where the <u>transition to the "double-logarithmic approximation"</u> ($\alpha_s Y \ln(x_T Q_{s0}) \sim 1$) occurs:

$$\boxed{k_{geom} \approx Q_{s0}\, e^{5.75\,\bar{\alpha}_s Y} = Q_s(Y)\cdot\left[\frac{Q_s(Y)}{Q_{s0}}\right]^{1.35}}$$

- Thus there is a <u>parametrically broad regime outside of saturation</u> where the <u>signature of saturation</u> is still present: $\left[Q_s(Y) \leq k_T \leq k_{geom}(Y)\right]$

<u>VIII</u>. <u>The Deep Saturation Limit</u>

In the $N \to 1$ asymptotic limit $S \equiv 1 - N$ becomes small, $S \ll 1$, and we can perform a <u>different linearization of the BK eqn:</u> (near Black Disk Limit)

$$\frac{\partial}{\partial Y}\left[1 - S(x_{10},Y)\right] = \frac{\alpha_s N_c}{2\pi^2}\int d^2x_2\, \frac{x_{10}^2}{x_{12}^2\, x_{20}^2}\Big[\big(\cancel{1} - \cancel{S(x_{12},Y)}\big) + \big(\cancel{1} - \cancel{S(x_{20},Y)}\big)$$

$$- \big(\cancel{1} - S(x_{10},Y)\big) - \underbrace{\big(1 - S(x_{12},Y)\big)\big(1 - S(x_{20},Y)\big)}_{\cancel{1} - \cancel{S(x_{12},Y)} - \cancel{S(x_{20},Y)}}\Big]$$

$$\hookrightarrow \boxed{\frac{\partial}{\partial Y} S(x_{10},Y) = \frac{-\alpha_s N_c}{2\pi^2}\left[\int d^2x_2\, \frac{x_{10}^2}{x_{12}^2\, x_{20}^2}\right] S(x_{10},Y)}$$

where the integral uses a Y-dependent cutoff $x_T \geq 1/Q_s(Y)$ in the UV.

$$\int_{x_T > 1/Q_s(Y)} d^2x_2 \, \frac{x_{10}^2}{x_{12}^2 \, x_{20}^2} = 4\pi \ln[x_{01} Q_s(Y)] \quad , \text{ so}$$

$$\frac{\partial}{\partial Y} S(x_{01}, Y) = -2 \underbrace{\frac{\alpha_s N_c}{\pi}}_{\bar{\alpha}_s} \ln[x_{01} Q_s(Y)] \cdot S(x_{01}, Y)$$

- Define the scaling variable

$$\boxed{\xi \equiv \ln[x_T^2 Q_s^2(Y)]}$$

so that

$$\frac{\partial \xi}{\partial Y} = 2 \frac{\partial}{\partial y} \ln Q_s(Y) = 2 \frac{\partial}{\partial Y}\left[\bar{\alpha}_s Y \frac{\chi(0, \gamma_0)}{1 + 2i\gamma_0}\right] = 2\bar{\alpha}_s \frac{\chi(0,\gamma_0)}{1+2i\gamma_0}$$

and

$$\frac{\partial}{\partial \xi} S(x_{01}, \xi) = \frac{\partial S}{\partial Y} \frac{\partial Y}{\partial \xi}$$

$$= \left[-\cancel{2\bar{\alpha}_s} \cdot \frac{\xi}{2} S(x_{01}, \xi)\right]\left[\frac{1 + 2i\gamma_0}{\cancel{2\bar{\alpha}_s} \, \chi(0, \gamma_0)}\right]$$

$$\frac{\partial}{\partial \xi} S(x_{01}, \xi) = -\frac{1 + 2i\gamma_0}{2\chi(0, \gamma_0)} \xi \cdot S(x_{01}, \xi)$$

which can be simply solved

$$\int_{S_0}^{S} \frac{dS}{S} = -\left(\frac{1+2i\gamma_0}{2\chi(0,\gamma_0)}\right) \int_{\xi=0}^{\xi} d\xi \cdot \xi$$

to give

$$\boxed{S = S_0 \exp\left[-\frac{1}{2}\left(\frac{1+2i\gamma_0}{2\chi(0,\gamma_0)}\right)\xi^2\right]}$$

or

$$\boxed{N(x_T, Y) = 1 - \exp\left[-\frac{1}{2}\left(\frac{1+2\nu_0}{2\chi(0,\nu_0)}\right)\ln^2\left(x_T^2 Q_s^2(Y)\right)\right]}$$

- Again, N is a function of a scaling variable ξ (or $\tau = x_T Q_s(Y)$), which is a sign that the physics is dominated by a single scale $Q_s(Y)$.

- This feature is known as "geometric scaling"

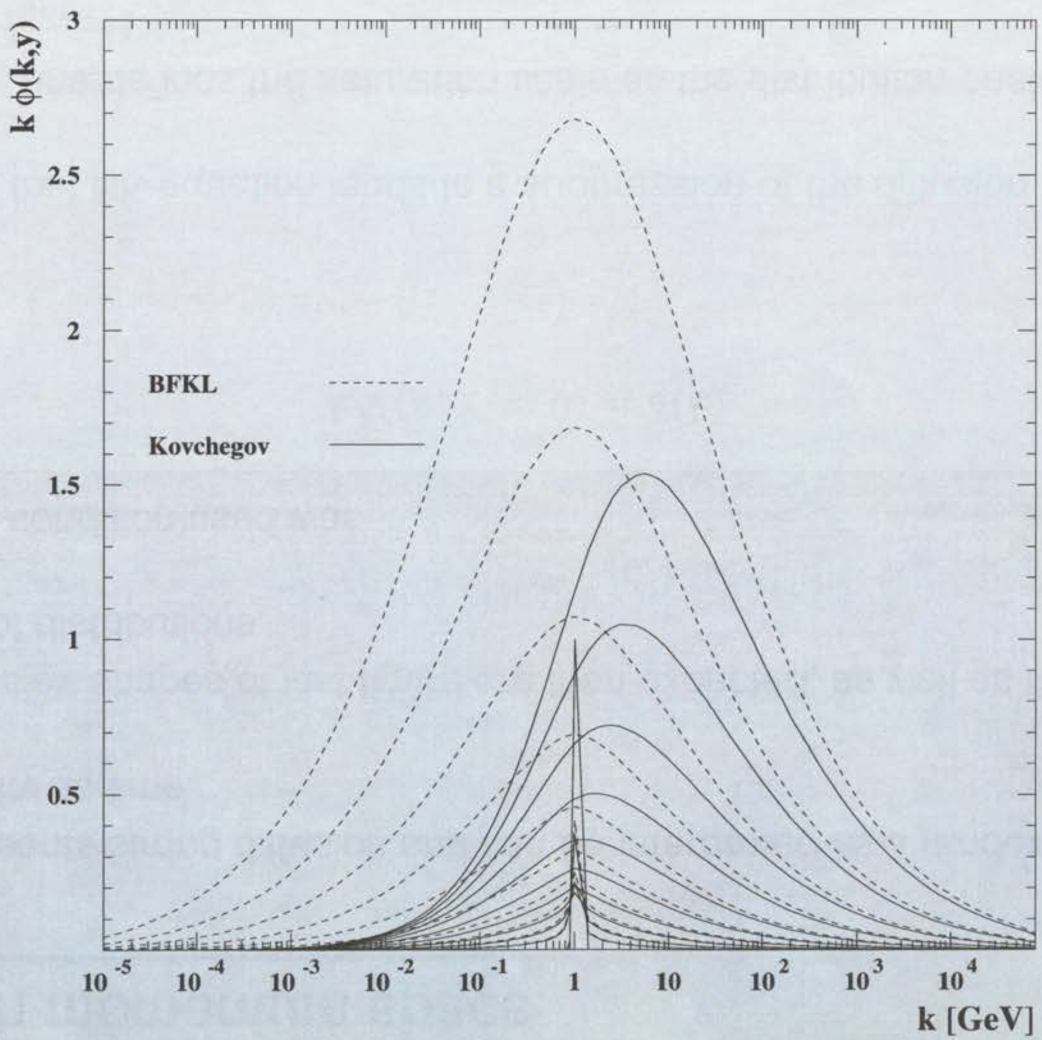
Because of the universality, many solution strategies will work, differing only in the functional form of the "knee."

IX. Phenomenology

- Numerical solution of the BK eqn verifies all these features:
  - Unitarization: $N \to 1$ as $Y \to \infty$ due to multiple rescattering
  - Geometric scaling seen in data at HERA
  - Cuts off diffusion into the IR, ensuring the validity of perturbation theory.

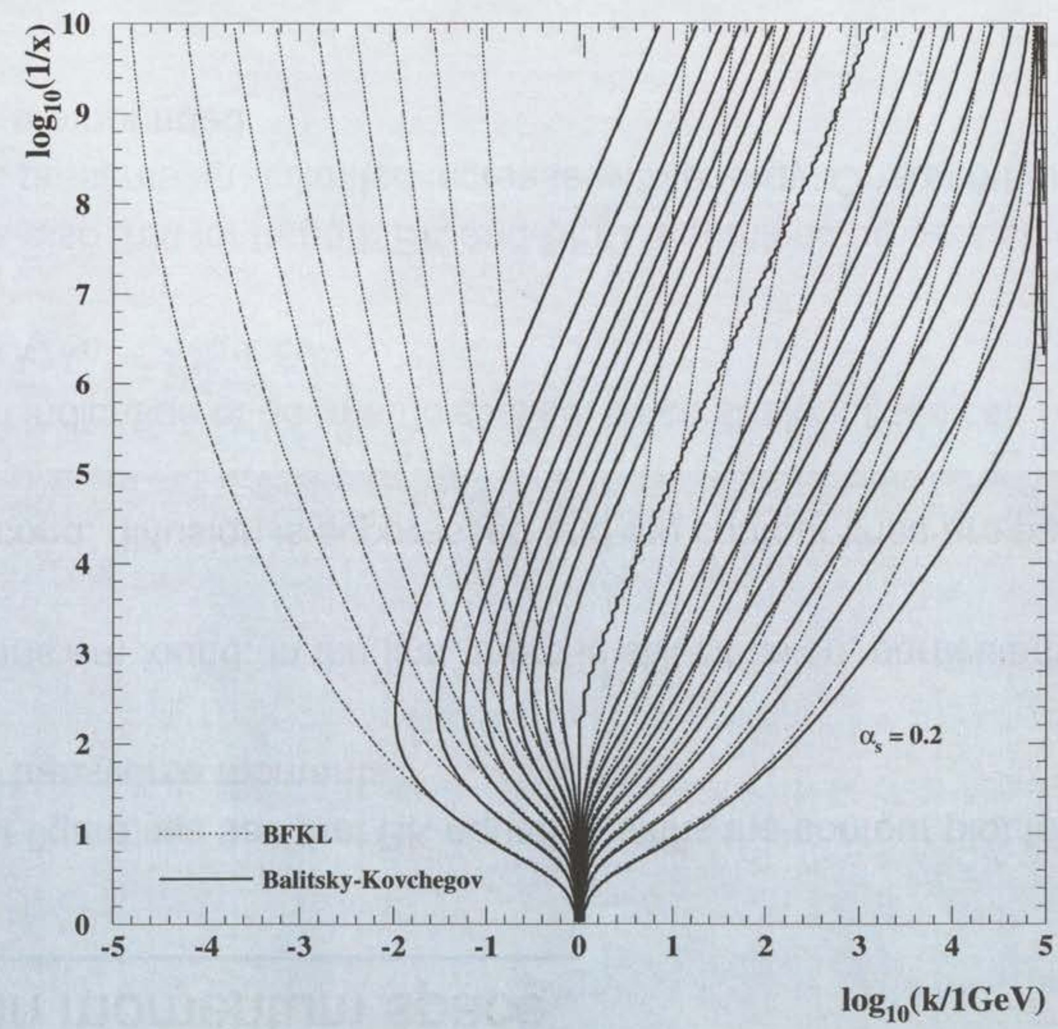
- The current "state of the art" tool for fitting experimental data is the BK eqn with running coupling corrections. This numerical "rcBK" solution matches data very well.

BK equation in momentum space



- Peak $\sim Q_s(Y)$ grows with Y in BK.
- Compared to BFKL, BK cuts off the diffusion of gluons into the IR.

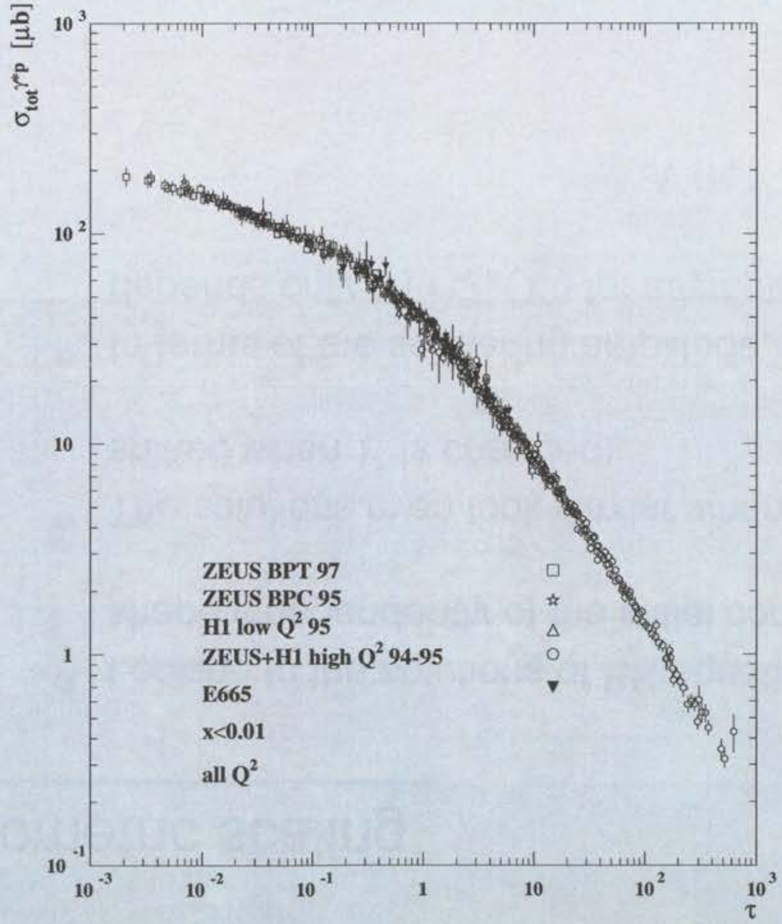
BK equation in momentum space



- BK moves into UV with increasing Y.
- Eliminates spread (diffusion) from BFKL, which would have drifted into the IR.

GFPAE Geometric scaling

- *Geometric scaling* is a phenomenological feature of DIS which has been observed in the HERA data on inclusive $\gamma^* - p$ scattering, which is expressed as a scaling property of the virtual photon-proton cross section

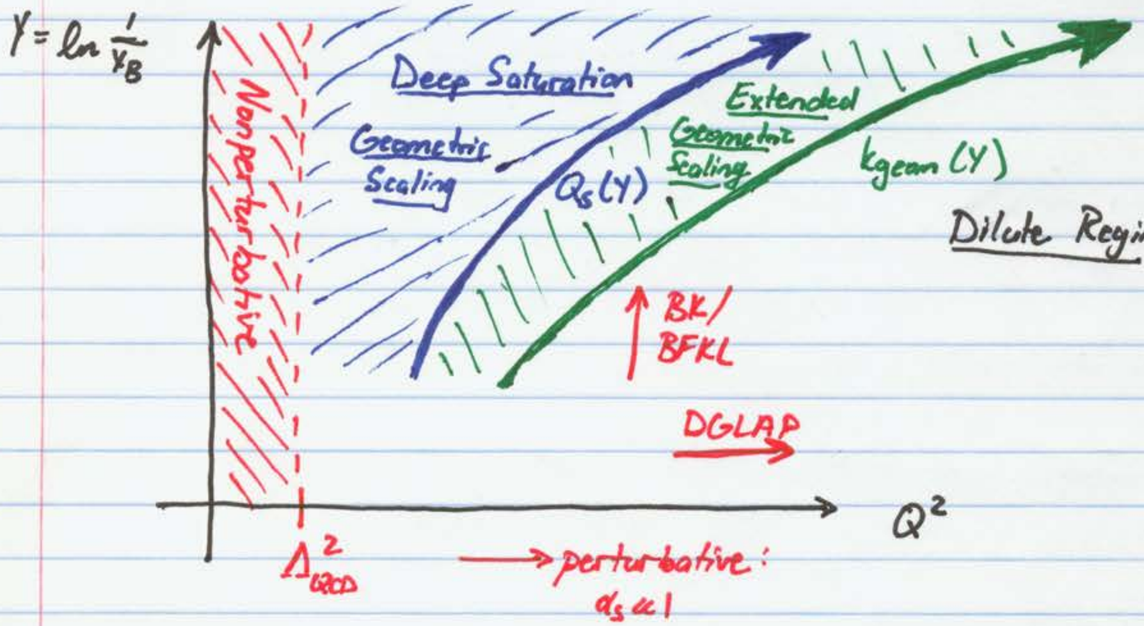


$$\sigma^{\gamma^* p}(Y,Q) = \sigma^{\gamma^* p}(\tau), \quad \tau = \frac{Q^2}{Q_s^2(Y)}$$

where Q is the virtuality of the photon, $Y = \log 1/x$ is the total rapidity and $Q_s(Y)$ is the saturation scale

[Stasto, Golec Biernat and Kwiecinsky, 2001]

Map of High Energy QCD:



- The BK eqn. corresponds to treating the nonlinear evolution by a mean-field approximation. While valid for a heavy nucleus, it is an approximation for a finite target.

- To go beyond BK, we need to include fluctuations and treat the whole n-gluon hierarchy without the large-N_c limit. This gives rise to a similar but more difficult evolution equation: the

  Jalilian-Marian
  Iancu
  McLerran
  Weigert
  Leonidov
  Korner

  } JIMWLK or "Jim Walk"

equation, which will be discussed next