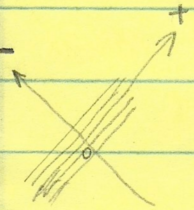


* Derivation of JIMWLK evolution

Frame: nucleus is moving along x^+-axis, while the projectile is moving along the x^--axis.

Gauge: $A^-=0$ light cone gauge of the projectile.

We define: $\alpha(x^-, \vec{x}_\perp) \equiv A^+(x^+=0, x^-, \vec{x}_\perp)$ for Compact Notation

with A^+ the fundamental-representation gluon field in $A^-=0$ gauge.

Yang-Mills equations.

$$\Box \alpha(x^-, \vec{x}_\perp) = \rho(x^-, \vec{x}_\perp)$$

$\alpha(x^-, \vec{x}_\perp)$ is related to $\rho(x^-, \vec{x}_\perp)$ and ρ_{LC}.

Defining a weight functional $W_Y[\alpha]$, then

$$\langle \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \; \hat{O}_\alpha W_Y[\alpha] \quad (*) \qquad \left(\int \mathcal{D}\alpha \, W_Y[\alpha] = 1 \right)$$

Goal: to construct an evolution equation for $W_Y[\alpha]$

Strategy:

1° derive an evolution equation for the expectation value of some test operator $\hat{O}_\alpha$

$$\partial_Y \langle \hat{O}_\alpha \rangle = \langle K_\alpha \otimes \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \, (\hat{K}_\alpha \otimes \hat{O}_\alpha) W_Y[\alpha]$$

K_α is the kernel of the equation.

2° Differentiating $\langle \hat{O}_\alpha \rangle_Y$ in $(*)$

$$\partial_Y \langle \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \, \hat{O}_\alpha \, \partial_Y W_Y[\alpha]$$

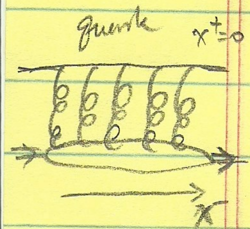
3°

$$\int \mathcal{D}\alpha \, \hat{O}_\alpha \partial_Y W_Y[\alpha] = \int \mathcal{D}\alpha \, (\hat{K}_\alpha \otimes \hat{O}_\alpha) W_Y[\alpha]$$

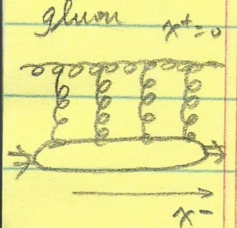
Test operator:

$$\hat{O}_{\vec{x}_{1\perp},\vec{x}_{0\perp}} = V_{\vec{x}_{1\perp}} \otimes V^{\dagger}_{\vec{x}_{0\perp}}$$

($V_{\vec{x}_{1\perp}}$: Wilson line for quark; $V^{\dagger}_{\vec{x}_{0\perp}}$: Wilson line for anti-quark; $\otimes$: color indices are fixed $\Rightarrow$ $= (V_{\vec{x}_{1\perp}})_{ij} (V^{\dagger}_{\vec{x}_{0\perp}})_{kl}$)



$$V_{\vec{x}_\perp} = P\exp\left\{\frac{ig}{2}\int_{-\infty}^{+\infty} dx^- \, t^a \alpha^a(x^-, \vec{x}_\perp)\right\}$$



Adjoint Wilson line

$$U_{\vec{x}_\perp} = P\exp\left\{\frac{ig}{2}\int_{-\infty}^{+\infty} dx^- \, T^a \alpha^a(x^-, \vec{x}_\perp)\right\}$$

We need to derive the evolution equation for $\hat{O}_{\vec{x}_{1\perp},\vec{x}_{0\perp}}$ in rapidity. The evolution is given by the long-lived s-channel gluons, which interact with traget over a relertively short peroid of time.

lifetime of s-channle gluon

$$x^-_{coh} = \frac{k^-}{k_\perp^2} \gg$$

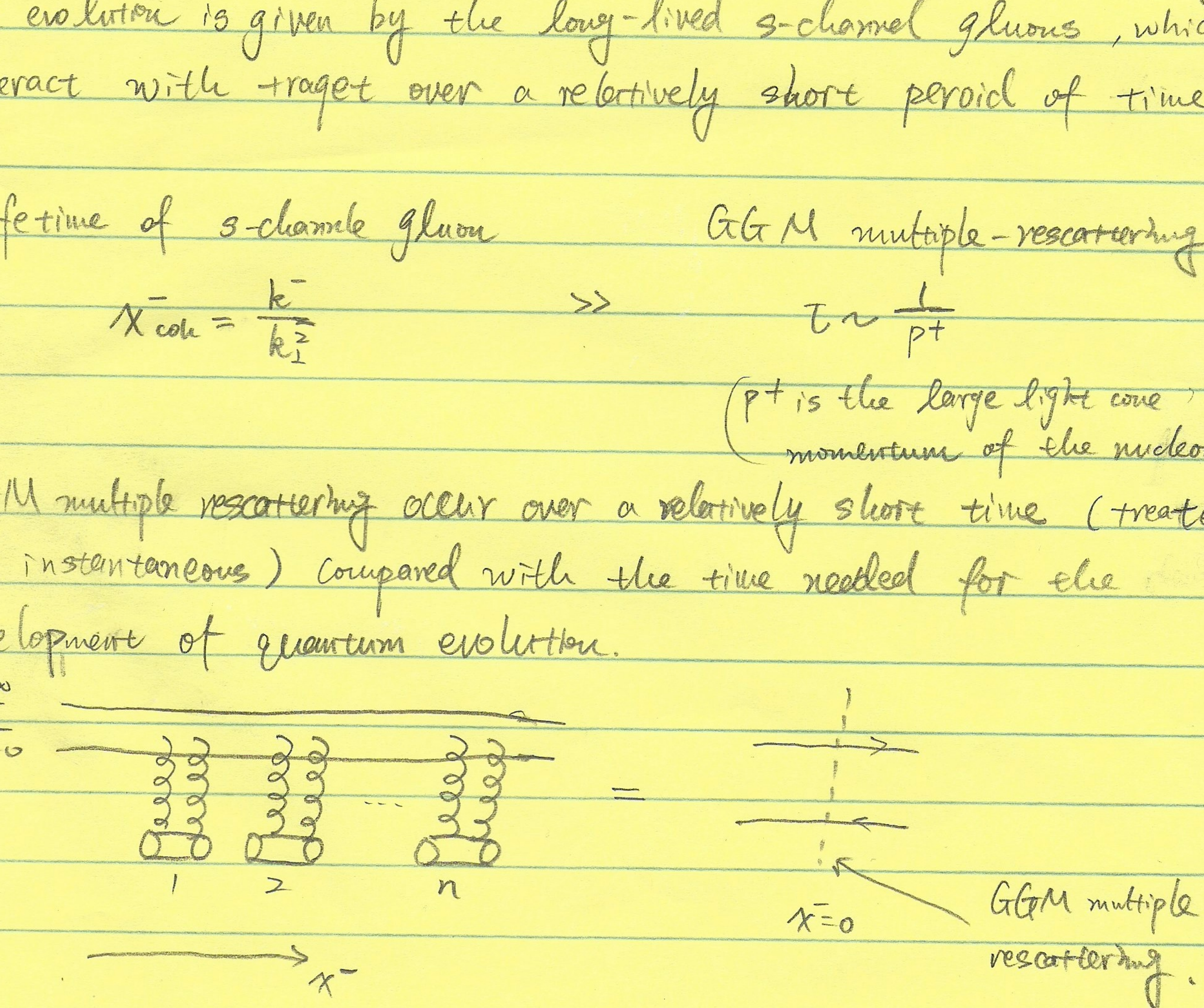
GGM multiple-rescattering

$$\tau \sim \frac{1}{p^+}$$

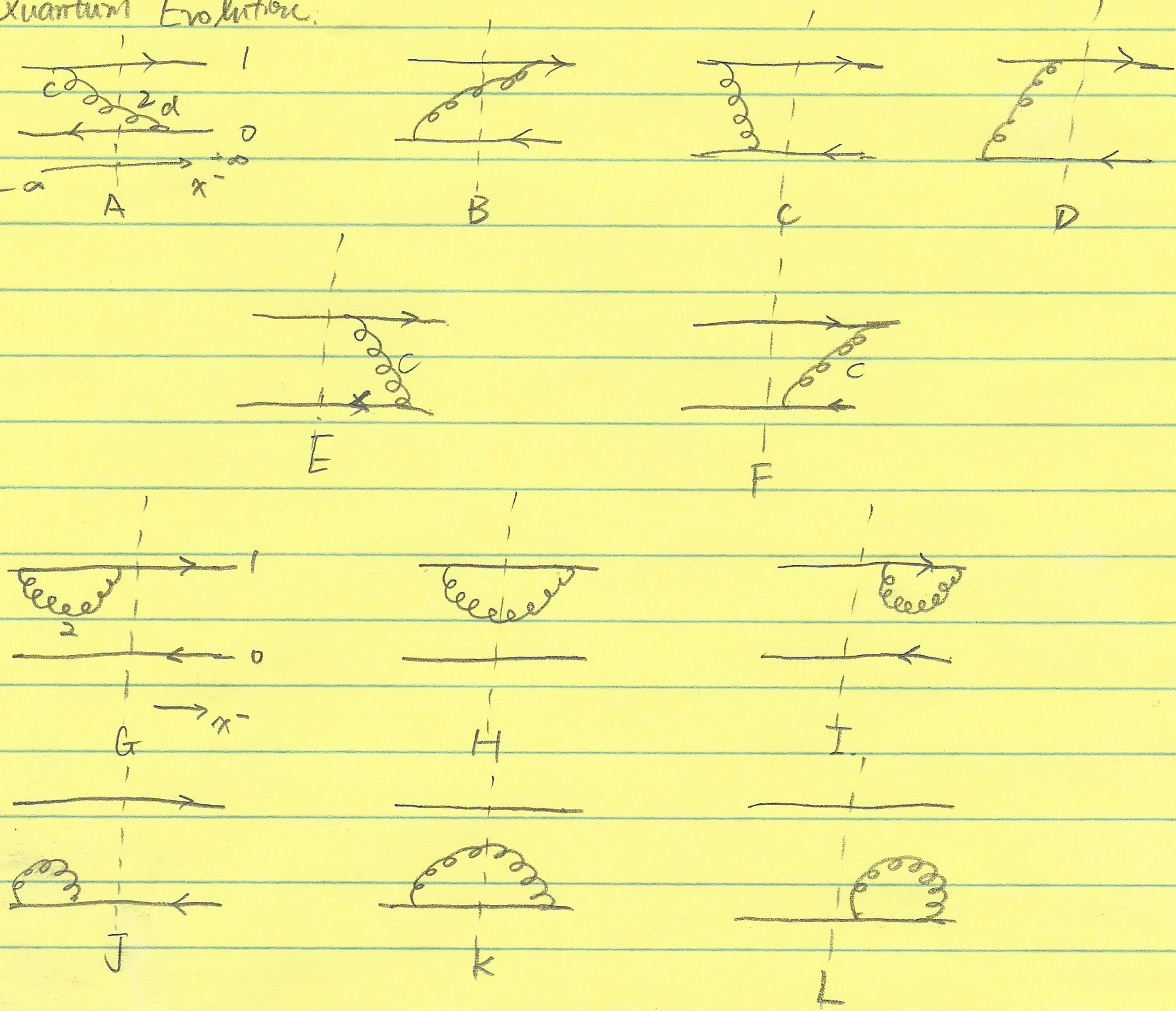
(p^+ is the large light cone momentum of the nucleon)

GGM multiple rescattering occur over a relatively short time (treated as instantaneous) compared with the time needed for the development of quantum evolution.

$$\sum_{n=0}^{\infty}$$



Quantum Evolution:



$$A + B + \cdots + F = \frac{\alpha_s}{\pi} \int d^2 x_2 \, dY \frac{\vec{x}_{21} \cdot \vec{x}_{20}}{x_{21}^2 x_{20}^2}$$

Resummed parameter $\alpha_s Y$

$$\times \left[1 - U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}} - U_{\vec{x}_{2\perp}} U^\dagger_{\vec{x}_{0\perp}} + U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{0\perp}} \right]_{ab} t^a V_{\vec{x}_{1\perp}} \otimes V^\dagger_{\vec{x}_{0\perp}} t^b$$

$$G + H + I = \frac{\alpha_s}{\pi} \int \frac{d^2 x_2}{x_{21}} dY \left[U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}} - 1 \right]_{ab} t^b t^a V_{\vec{x}_{1\perp}} \otimes V^\dagger_{\vec{x}_{0\perp}}$$

$$J + K + L = \frac{\alpha_s}{\pi} \int \frac{d^2 x_2}{x_{20}^2} dY \left[U_{\vec{x}_{2\perp}} U^\dagger_{\vec{x}_{0\perp}} - 1 \right]_{ab} V_{\vec{x}_{1\perp}} \otimes V^\dagger_{\vec{x}_{0\perp}} t^b t^a$$

In order to derive JIMWLK, we can write them in terms of functional derivatives of $\alpha^a(x^-, \vec{x}_\perp)$,

$$A+B+\cdots+F = \frac{\alpha_s}{2}\int d^2x_\perp\, d^2y_\perp\, dY\, \eta^{ab}_{\vec{x}_\perp, \vec{y}_\perp} \frac{\delta^2 (V_{\vec{x}_{1\perp}} \otimes V^\dagger_{\vec{x}_{0\perp}})}{\delta\alpha^a(x^-, \vec{x}_\perp)\, \delta\alpha^b(y^-, \vec{y}_\perp)}$$

with

$$\eta^{ab}_{\vec{x}_\perp, \vec{x}_{0\perp}} = \frac{4}{g^2\pi^2}\int d^2x_2 \frac{\vec{x}_{21}\cdot\vec{x}_{20}}{x_{21}^2\, x_{20}^2} \left[\mathbb{1} - U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}} - U_{\vec{x}_{2\perp}} U^\dagger_{\vec{x}_{0\perp}} + U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{0\perp}} \right]^{ab}$$

for $x^-, y^- > 0$. One functional derivative acts on V and the other acts on $V^\dagger$

$$G+H+I = \frac{\alpha_s}{2}\int d^2x_\perp\, d^2y_\perp\, dY\, \eta^{ab}_{\vec{x}_\perp, \vec{y}_\perp} \left[\frac{\delta^2 V_{\vec{x}_{1\perp}}}{\delta\alpha^a(x^-, \vec{x}_\perp)\, \delta\alpha^b(y^-, \vec{y}_\perp)} \right] \otimes V^\dagger_{\vec{x}_{0\perp}}$$

$$+ \alpha_s \int d^2x_\perp\, \nu^a_{\vec{x}_\perp} \left[\frac{\delta V_{\vec{x}_{1\perp}}}{\delta\alpha^a(x^-, \vec{x}_\perp)} \right] \otimes V^\dagger_{\vec{x}_{0\perp}}$$

$$J+K+L = \frac{\alpha_s}{2}\int d^2x_\perp\, d^2y_\perp\, dY\, \eta^{ab}_{\vec{x}_\perp, \vec{y}_\perp}\, V_{\vec{x}_{1\perp}} \otimes \left[\frac{\delta^2 V^\dagger_{\vec{x}_{0\perp}}}{\delta\alpha^a(x^-, \vec{x}_\perp)\, \delta\alpha^b(y^-, \vec{y}_\perp)} \right]$$

$$+ \alpha_s \int d^2x_\perp\, \nu^a_{\vec{x}_\perp}\, V_{\vec{x}_{1\perp}} \otimes \left[\frac{\delta V^\dagger_{\vec{x}_{0\perp}}}{\delta\alpha^a(x^-, \vec{x}_\perp)} \right]$$

where $\nu^a_{\vec{x}_{1\perp}} = \frac{i}{g\pi^2}\int \frac{d^2x_2}{x_{21}^2} \mathrm{Tr}\left[T^a U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}} \right]$

with $x^-, y^- > 0$.

$$\therefore\ \partial_Y \langle \hat{O}_{\vec{x}_{1\perp}, \vec{x}_{0\perp}} \rangle_Y = \frac{\alpha_s}{2}\int d^2x_\perp\, d^2y_\perp \left\langle \eta^{ab}_{\vec{x}_\perp, \vec{y}_\perp} \frac{\delta^2 \hat{O}_{\vec{x}_{1\perp}, \vec{x}_{0\perp}}}{\delta\alpha^a(x^-, \vec{x}_\perp)\, \delta\alpha^b(y^-, \vec{y}_\perp)} \right\rangle_Y$$

$$+ \alpha_s \int d^2x_\perp \left\langle \nu^a_{\vec{x}_\perp} \frac{\delta \hat{O}_{\vec{x}_{1\perp}, \vec{x}_{0\perp}}}{\delta\alpha^a(x^-, \vec{x}_\perp)} \right\rangle_Y$$

We have $\langle \hat{O} \rangle_Y = \int \mathcal{D}\alpha\, \hat{O}_\alpha\, W_Y[\alpha]$. Using integration by part on the right hand side of the equation, we can get

$$\int \mathcal{D}\alpha \, \hat{O}_{\vec{x}_{1\perp}, \vec{x}_{0\perp}} \partial_Y W_Y[\alpha] = \int \mathcal{D}\alpha \, \hat{O}_{\vec{x}_{1\perp}, \vec{x}_{0\perp}} \left\{ \frac{\alpha_s}{2} \int d^2x_\perp d^2y_\perp \frac{\delta^2}{\delta\alpha^a(x^-, \vec{x}_\perp)\delta\alpha^b(y^-, \vec{y}_\perp)} \left(\eta^{ab}_{\vec{x}_\perp, \vec{y}_\perp} W_Y[\alpha] \right) \right.$$

$$\left. - \alpha_s \int d^2x_\perp \frac{\delta}{\delta\alpha^a(x^-, \vec{x}_\perp)} \left(\nu^a_{\vec{x}_\perp} W_Y[\alpha] \right) \right\} .$$

↑ comes from integration by part

∴ JIMWLK equation

$$\partial_Y W_Y[\alpha] = \frac{\alpha_s}{2} \int d^2x_\perp d^2y_\perp \frac{\delta^2}{\delta\alpha^a(x^-, \vec{x}_\perp)\delta\alpha^b(y^-, \vec{y}_\perp)} \left(\eta^{ab}_{\vec{x}_\perp, \vec{y}_\perp} W_Y[\alpha] \right)$$
$$- \alpha_s \int d^2x_\perp \frac{\delta}{\delta\alpha^a(x^-, \vec{x}_\perp)} \left(\nu^a_{\vec{x}_\perp} W_Y[\alpha] \right)$$

This is a differential equation for the weight functional $W_Y[\alpha]$. The Gaussian form of the functional (from MV model) serves as its initial condition.

This equation resums all powers of $\alpha_s Y$ and Gaussian initial condition resums all classical physic effects ($\alpha_s A^{1/3}$)

For operator $\hat{O}$ constructed from the fundamental or adjoint Wilson lines, the JIMWLK evolution for its expectation value is

$$\partial_Y \langle \hat{O} \rangle_Y = \frac{\alpha_s}{2} \int d^2x_\perp d^2y_\perp \left\langle \eta^{ab}_{\vec{x}_\perp, \vec{y}_\perp} \frac{\delta^2 \hat{O}}{\delta\alpha^a(x^-, \vec{x}_\perp)\delta\alpha^b(y^-, \vec{y}_\perp)} \right\rangle_Y$$
$$+ \alpha_s \int d^2x_\perp \left\langle \nu^a_{\vec{x}_\perp} \frac{\delta \hat{O}}{\delta\alpha^a(x^-, \vec{x}_\perp)} \right\rangle_Y$$

* Obtaining BK from JIMWLK and Balitsky hierarchy

Define the S-matrix operator,

$$\hat{S}_{\vec{x}_{1\perp},\vec{x}_{0\perp}} = \frac{1}{N_c} \mathrm{tr}\left[V_{\vec{x}_{1\perp}} V^{\dagger}_{\vec{x}_{0\perp}} \right].$$

The S-matrix is then given by

$$S(\vec{x}_{1\perp}, \vec{x}_{0\perp}, Y) = \langle \hat{S}_{\vec{x}_{1\perp},\vec{x}_{0\perp}} \rangle_Y .$$

Put this operator into JIMWLK equation, one get

$$\partial_Y \langle \hat{S}_{\vec{x}_{1\perp},\vec{x}_{0\perp}} \rangle_Y = \frac{\alpha_s}{2} \int d^2x_\perp d^2y_\perp \left\langle \eta^{ab}_{\vec{x}_\perp,\vec{y}_\perp} \frac{\delta^2 \hat{S}_{\vec{x}_{1\perp},\vec{x}_{0\perp}}}{\delta\alpha^a(x^-,\vec{x}_\perp)\,\delta\alpha^b(y^-,\vec{y}_\perp)} \right\rangle_Y$$

$$+ \alpha_s \int d^2x_\perp \left\langle \nu^a_{\vec{x}_\perp} \frac{\delta \hat{S}_{\vec{x}_{1\perp},\vec{x}_{0\perp}}}{\delta\alpha^a(x^-,\vec{x}_\perp)} \right\rangle_Y$$

After large mount of algebra, one obtains

$$\partial_Y \langle \hat{S}_{\vec{x}_{1\perp},\vec{x}_{0\perp}} \rangle_Y = \frac{\bar{\alpha}_s}{2\pi} \int d^2x_2 \frac{x_{10}^2}{x_{20}^2 x_{21}^2} \left[\langle \hat{S}_{\vec{x}_{1\perp},\vec{x}_{2\perp}} \hat{S}_{\vec{x}_{2\perp},\vec{x}_{0\perp}} \rangle_Y - \langle \hat{S}_{\vec{x}_{1\perp},\vec{x}_{0\perp}} \rangle_Y \right] \quad (*)$$

Large N_c limit $\Rightarrow$ If

$$\langle \hat{S}_{\vec{x}_{1\perp},\vec{x}_{2\perp}} \hat{S}_{\vec{x}_{2\perp},\vec{x}_{0\perp}} \rangle_Y \to \langle \hat{S}_{\vec{x}_{1\perp},\vec{x}_{2\perp}} \rangle_Y \langle \hat{S}_{\vec{x}_{2\perp},\vec{x}_{0\perp}} \rangle_Y$$

it reduces to BK equation, which is a closed equation for $\langle \hat{S}_{\vec{x}_{1\perp},\vec{x}_{2\perp}} \rangle_Y$.

Without taking large N_c limit. (*) is not a closed equation. It depends on a new four-Wilson-line operator

$$\langle \hat{S}_{\vec{x}_{1\perp},\vec{x}_{2\perp}} \hat{S}_{\vec{x}_{2\perp},\vec{x}_{0\perp}} \rangle_Y .$$

Its evolution contains an operator with six fundamental Wilson lines.

$$\langle \hat{S}\hat{S}\hat{S} \rangle$$

So the evolution of the n-Wilson-line operator would be driven by an (n+2)-Wilson-line operator. This is called the Balitsky hierarchy. BK equation truncates the Balitsky hierarchy at lowest order.

importance of $\frac{1}{N_c}$ correction

Numerical solution of full JIMWLK equation is compared with solutions from BK equation. Due to saturation effects, $\frac{1}{N_c}$ corrections to $\langle \hat{S}_{\vec{x}_{1\perp}, \vec{x}_{0\perp}} \rangle_Y$ is suppressed. The difference is only $\sim 0.1\%$.