McLerran-Venugopalan Model.

(Quasi-Classical Approximation.)

Let us first pose some problems related to the problems of BFKL evolution. Once we solve them, we'll know how to deal with BFKL problems as well.

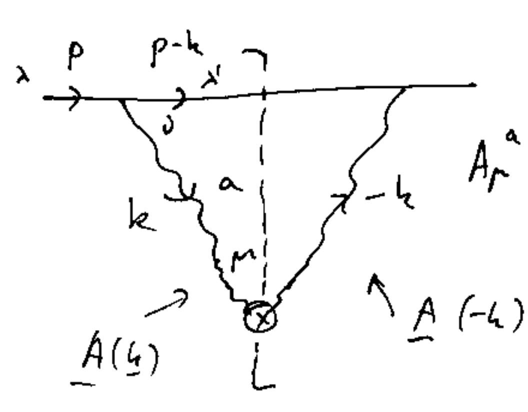
$\Rightarrow$ In light-cone gauge, $A_+ = 0$, we argued that gluon distribution is $\quad x\, G(x, Q^2) \sim \int^{Q^2} d^2 k_T \underbrace{\langle A_i(-k) A_i(k) \rangle}_{k_T^2}$ transverse only, $A_+ = 0$

(like $a_k^\dagger a_k$ ~ particle number operator)

We therefore define unintegrated gluon distr.

$$\varphi(x, k_T^2) = \frac{k_T^2}{(2\pi)^2} \langle \underline{A}^a(-k) \cdot \underline{A}^a(k) \rangle$$

$\Rightarrow$ Let's calculate $\varphi(x, k_T^2)$ of a single quark at the lowest order:



Calculate one field

$$A_\mu^a = \underbrace{\frac{1}{2p_+}}_{\text{flux}} ig\, \bar{u}_{\lambda'}(p-k)\, \gamma_\nu\, u_\lambda(p) \cdot \frac{-i}{k^2} \cdot \left[g_{\mu\nu} - \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{k_+} \right] T^a$$

0

as $\bar{u}(p-k)\, \not{k}\, u(p) = \bar{u}(p-k) [-(\not{p} - \not{k}) + \not{p}] u(p) = 0$

since $\not{p}\, u(p) = 0$, $(\not{p} - \not{k})\, u(p-k) = 0$ (Dirac eqns.)

$$A_\mu^a = g\, T^a \frac{1}{k^2}\, \frac{1}{2p_+} \Big[\underbrace{\tilde{u}_{\lambda'}(p-k)\, \gamma_\mu\, u_\lambda(p)}_{2p_+ \delta_{\mu +}\, \delta_{\lambda\lambda'}} - \frac{k_\mu}{k_+} \underbrace{\tilde{u}_{\lambda'}(p-k)\, \gamma_+\, u_\lambda(p)}_{2p_+\, \delta_{\lambda\lambda'}} \Big] =$$

biggest contribution if p_+ is large (see Brodsky & Lepage handout)

$$= g\, T^a \frac{1}{k^2}\, \delta_{\lambda\lambda'} \Big[\delta_{\mu +} - \frac{k_\mu}{k_+} \Big] = - g\, T^a \frac{1}{\underline{k}^2}\, \delta_{\lambda\lambda'} \cdot \frac{k_\mu^\perp}{k_+}$$

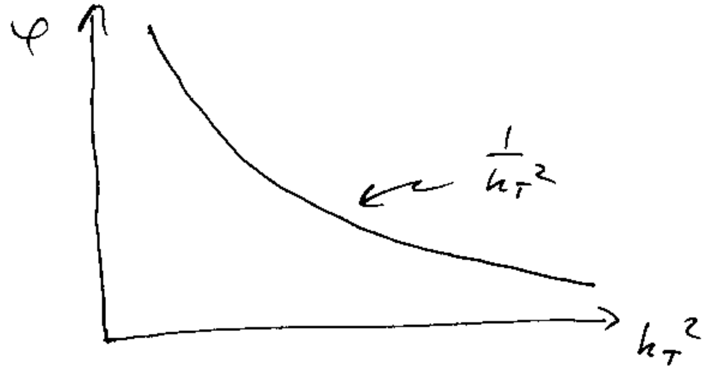
as $(p-k)^2 = 0 \Rightarrow 2p_+ k_- \simeq 0 \Rightarrow k_- \simeq 0. \Rightarrow k^2 = 2k_+k_- - \underline{k}^2 \simeq$

$\simeq - \underline{k}^2$.

$$\langle A_i^a(-\underline{k}) \cdot A_i^a(\underline{k}) \rangle = g^2 \overbrace{(T^a T^a)}^{C_F = \frac{N_c^2-1}{2N_c}} \underbrace{\frac{1}{\underline{k}^4}\, k_i \cdot k_i}_{1/\underline{k}^2} \frac{1}{k_+^2} =$$

$$= 4\pi \alpha_s C_F \frac{1}{\underline{k}^2} \frac{1}{k_+^2} \Rightarrow \quad \varphi(x, k_T^2) = \frac{\alpha\, C_F}{\pi} \frac{1}{\underline{k}^2}$$

(we summed over final and averaged over initial gluon helicities)



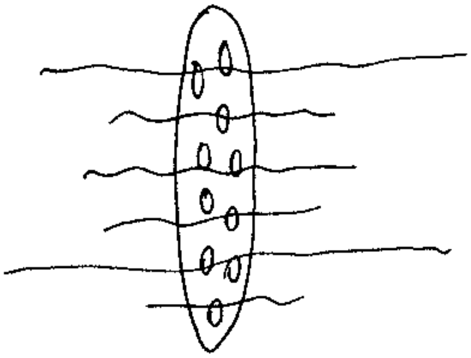
$\Rightarrow$ a new problem arises:

$$\int_\Lambda^Q d^2k_T\, \varphi \sim \int_\Lambda^Q \frac{dk_T^2}{k_T^2} = \ln \frac{Q^2}{\Lambda^2}$$

as $\Lambda \to 0 \Rightarrow$ get ∞ number of gluons !?

(or, $A_\mu \lesssim \frac{Q}{g}$: but $A_\mu \sim \frac{g}{k_T} \to \infty$ as $k_T \to 0$)

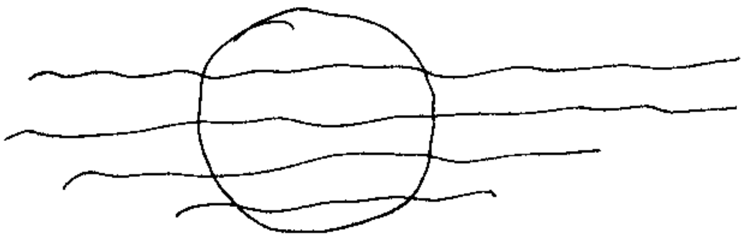
NOT TO mention that $\alpha_s(\Lambda_{QCD}) \gtrsim 1 \ldots$



Imagine an ultrarelativistic nucleus. Its wave function has many small-x gluons: $x = \frac{k_+}{p_+}$

$$\Delta x_- \simeq \frac{1}{k_+} = \frac{1}{x p_+}$$

In the rest frame of the nucleus: $p_+ = \frac{m_N}{\sqrt{2}}$



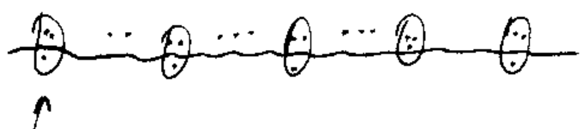
$$l_{coh} \sim \frac{1}{x p_+} \sim \frac{1}{x m_N}$$

$\Rightarrow$ small-x leads to large l_{coh}

(e.g. $x = 10^{-3} \Rightarrow l_{coh} \simeq \frac{1}{10^{-3} \cdot 1\,\text{GeV}} = 10^3 \cdot .2\,\text{fm} = 200\,\text{fm}$

$x = 10^{-4} \Rightarrow l_{coh} \simeq 2000\,\text{fm}$!)

$\Rightarrow$ Small-x gluons are coherent throughout the nucleus! Can interact with many nucleons:



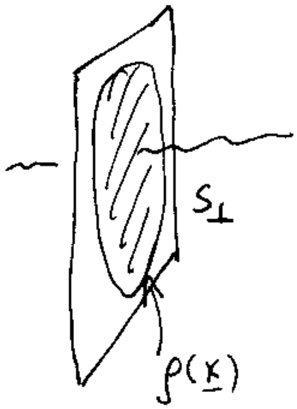
$= \Delta x_\perp \sim \frac{1}{k_T}$ – small, don't see the whole nucleon

gluon sees color charge

Total color charge seen by a gluon is $Q = \sqrt{N}\, g$ ← strong coupling

No. of nucleons, $N \sim A$ (random walk)

Back in the infinite momentum frame



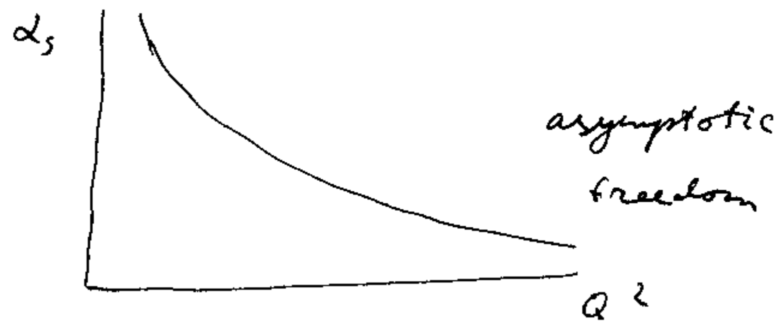
density of color charge

$$\mu^2 = \frac{Q^2}{S_\perp} \propto \frac{A g^2}{\pi R^2} \sim A^{1/3} \Lambda^2_{QCD}$$

gets <u>large</u> for large nuclei.

$\Rightarrow$ Large momentum scale $\Rightarrow$ small coupling

$$\boxed{\alpha_s(\mu^2) \ll 1}$$



$\Rightarrow$ Small coupling ~ classical fields dominate!

(quantum corrections come as loops ~ higher orders in α_s)

$\Rightarrow$ Alternatively, we have high occupation numbers of color charges, get large Q (higher-dim represent.)

such that $[\hat{Q}_i, \hat{Q}_j] \simeq 0$ (can neglect the commutators)

$\hookrightarrow$ gluon field is classical!

Need to solve classical Yang-Mills equation of motion

$$\boxed{\mathcal{D}_\mu F^{\mu\nu} = J^\nu}$$

or

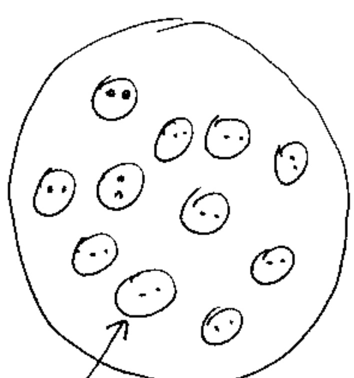
$$\partial_\mu F^{\mu\nu} - ig\,[A_\mu, F^{\mu\nu}] = J^\nu$$

with $J^\nu = \delta^{\nu +} \overbrace{\delta(x_-)\,\rho(\underline{x})}^{\rho(x_-,\underline{x})}$ (McLerran Venugopalan '93-'94)

$\uparrow$ color charge density

We need to find solution in $A_+ = 0$ light-cone gauge, since $\varphi(x, k_T^2) = \frac{k_\perp^2}{(2\pi)^2} \langle \underline{A}^a(-k)\cdot \underline{A}^a(k)\rangle$ only in LC gauge.

Let us start by constructing an explicit model of the nucleus: imagine a nucleus made of quarkonia:



working in covariant gauge $\partial_\mu A^\mu = 0$ (more convenient, btw current J is gauge-dependent!)

write

$$\rho^a(\underline{x}, x_-) = g \sum_{i=1}^{N} (T^a_i)\left[\delta(x_- - x_{i-})\,\delta(\underline{x} - \underline{x}_i) - \delta(x_- - x'_{i-})\,\delta(\underline{x} - \underline{x}'_i)\right]$$

ith nucleon

quark is at $\underline{x}_i, x_{i-}$

anti-quark is at $\underline{x}'_i, x'_{i-}$

Working in $\partial_\mu A^\mu = 0$ gauge let us start by constructing the field of a point UR charge e in QED: massaging Lienard-Wiechert potentials we get

$e^{\bullet}\ (\underline{x}=0,\ x_-=0)$

$\rho' = e\,\delta(x_-)\,\delta(\underline{x})$

$$\begin{cases} A'_+ = -\dfrac{e}{2\pi}\,\delta(x^-)\,\ln(|\underline{x}|\Lambda) \\ \underline{A}'=0,\ A'_-=0 \end{cases}$$

Let's check that this is a solution of maxwell equations $\partial_\mu F^{\mu\nu} = \delta^{\nu+}\, e\,\delta(x_-)\,\delta(\underline{x})$

$$F'_{+i} = -\partial_i A'_+ = -\frac{e}{2\pi}\,\delta(x_-)\,\frac{\underline{x}_i}{\underline{x}^2}\ ,\ F'_{+-}=0,\ F'_{ij}=0,\ F'_{-i}=0.$$

$$\partial_\mu F'^{\mu+} \Rightarrow \partial_+ \overset{=0}{F'_{-+}} + \partial_- \overset{=0}{F'_{++}} - \partial_i F'_{i+} = e\,\delta(x_-)\,\delta(\underline{x})$$

as $\partial_i^2 \ln(|\underline{x}|\Lambda) = 2\pi\,\delta(\underline{x})$; $\partial_\mu F'_{\mu i} = \partial_- F'_{+i} = \dfrac{\partial}{\partial x_+} F'_{+i} = 0.$

Now, let us generalize it to a <u>color</u> charge

$g^{\bullet}\,T^a\ (\underline{x}=0,\ x_-=0)$ in $\partial_\mu A^\mu = 0$ covariant gauge

get
$$\begin{cases} A^{a\prime}_+ = -\dfrac{g}{2\pi}\,T^a\,\delta(x^-)\,\ln(|\underline{x}|\Lambda), \\ \underline{A}^{a\prime}=0,\ A^{a\prime}_-=0 \end{cases}$$

Again, need to check that $\mathcal{D}_\mu F^{a\mu\nu} = g\,T^a\cdot\delta(x_-)\,\delta(\underline{x})$

$$F^{a\prime}_{+i} = -\frac{g}{2\pi}\,T^a\,\delta(x_-)\,\frac{x_i}{\underline{x}^2} \Rightarrow \partial_\mu F^{a\prime}_{\mu+} + g f^{abc} A^{b\prime}_\mu F^{c\prime}_{\mu+} =$$

$$= g\,T^a\,\delta(x_-)\,\delta(\underline{x}) + g f^{abc}\cdot(-)\,\overset{=0}{A'^{b}_i}\,F'^{c}_{i+} \quad \text{ok.}$$

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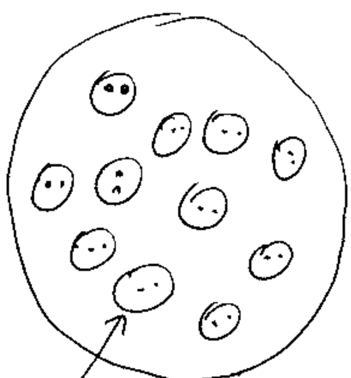

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quark is at $\underline{x}_i, x_{i-}$

anti-quark is at $\underline{x}_i', x_{i-}'$

working in covariant gauge $\partial_\mu A^\mu = 0$ (more convenient, btw current J is gauge-dependent!)

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$$\rho^a(\underline{x}, x_-) = g \sum_{i=1}^{N} (T_i^a) \left[\delta(x_- - x_{i-})\, \delta(\underline{x} - \underline{x}_i) - \delta(x_- - x_{i-}')\, \delta(\underline{x} - \underline{x}_i') \right]$$

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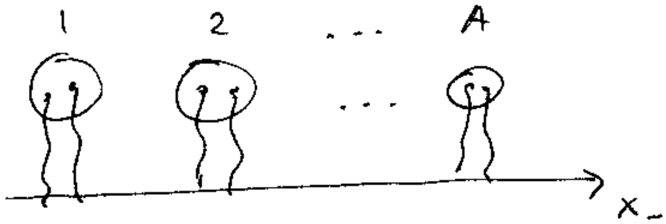


$$x'_{1-} < x'_{2-} \qquad x'_{A-}$$

$$X_{1-} < X_{2-} < \ldots < X_{A-}$$

(same for x'_{i-})

Now, fields of individual point charges $\propto \delta(x_-)$

$\Rightarrow$ never overlap $\Rightarrow$ superposition is OK!

$$\begin{cases} A_+^{a\prime} = -\dfrac{g}{2\pi} \displaystyle\sum_{i=1}^{A} (T_i^a) \left[\delta(x_- - x_{i-}) \ln(|\underline{x} - \underline{x}_i|\Lambda) - \delta(x_- - x'_{i-}) \ln(|\underline{x} - \underline{x}'_i|\Lambda) \right] \\ \underline{A}'^a = 0, \quad A'^a_- = 0. \end{cases}$$

Need to find field in $A_+ = 0$ LC gauge

$$A_\mu = S A'_\mu S^{-1} - \frac{i}{g} (\partial_\mu S) S^{-1}$$

such that $A_+ = 0$:

$$A_+ = S A'_+ S^{-1} - \frac{i}{g} (\partial_+ S) S^{-1} = 0 \Rightarrow \partial_+ S = -ig S A'_+$$

$$\Rightarrow S(\underline{x}, x_-) = P \exp\left\{ -ig \int_{-\infty}^{x_-} dx'_- \, A'_+(\underline{x}, x'_-) \right\} \Rightarrow$$

$$\boxed{S(\underline{x}, x_-) = \prod_{i=1}^{A} \exp\left[\frac{i g^2}{2\pi} \sum_{a=1}^{N_c^2-1} T^a (T_i^a) \ln\left(\frac{|\underline{x} - \underline{x}_i|}{|\underline{x} - \underline{x}'_i|} \right) \theta(x_- - x_{-i}) \right]}$$

Finally, to find the LC gauge field use

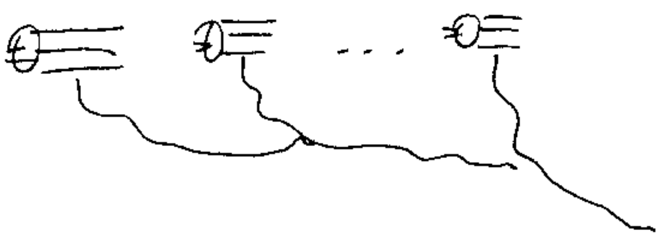
$$\underset{\partial_+ A_i}{\underset{\shortparallel}{F_{+i}}} = S F'_{+i} S^{-1} \Rightarrow A_i = \int_{-\infty}^{x_-} dx'_- \, S(\underline{x}, x'_-) \, F'_{+i}(\underline{x}, x'_-) \, S^{-1}(\underline{x}, x'_-)$$

Finally,

$$\boxed{\underline{A}(\underline{x}, x_-) = \frac{g}{2\pi} \sum_{a=1}^{N_c^2-1} \sum_{i=1}^{A} (T_i^a) \left\{ S(\underline{x}, x_{i-})\, T^a\, S^{-1}(\underline{x}, x_{i-}) \frac{\underline{x}-\underline{x}_i}{|\underline{x}-\underline{x}_i|^2} \cdot \theta(x_- - x_{i-}) - S(\underline{x}, x'_{i-})\, T^a\, S^{-1}(\underline{x}, x'_{i-}) \frac{\underline{x}-\underline{x}'_i}{|\underline{x}-\underline{x}'_i|^2}\, \theta(x_- - x'_{i-}) \right\}}$$

this is the non-Abelian Weizsäcker-Williams field of a large nucleus. (Yu. K. '96, Jalilian-Marian et al '96)

Diagrammatically it corresponds to



Classical fields $\Leftrightarrow$ tree diagrams

If we have a Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + J_\mu^a A^{a\mu} = \frac{1}{2} A_\mu^a \underbrace{\left[\mathcal{D}_{\mu\nu}^{ab}\right]^{-1}}_{\text{inverse propagator}} A_\nu^b + g f^{abc} A_\mu^a A_\nu^b \partial_\mu A_\nu^c + \ldots$$

$$+ \frac{g^2}{4} f^{abe} f^{cde} A_\mu^a A_\nu^b A_\mu^c A_\nu^d + J_\mu^a A^{a\mu}.$$

To get equations of motion require that $\frac{\delta \mathcal{L}}{\delta A_\mu^a} = 0$

One gets

$$[\mathcal{D}_{\mu\nu}^{ab}]^{-1} A_\nu^b = g f^{abc} A_\nu^b \partial_\mu A_\nu^c + \frac{g^2}{4} f^{abc} f^{cde} A_\nu^b A_\mu^c A_\nu^d \, J_\mu^a$$

Let's try solving it perturbatively:

start with weak field A & source J:

if A is small, we neglect all higher powers of A:

$$[\mathcal{D}_{\mu\nu}^{ab}]^{-1} A_\nu^{(1)b} = J_\mu^{(1)a} \Rightarrow \boxed{A_\mu^{(1)a} = -[\mathcal{D}_{\mu\nu}^{ab}] \cdot J_\nu^{(1)b}}$$

propagator

To go to next order need to impose current conservation: $\mathcal{D}_\mu F^{\mu\nu} = J^\nu \Rightarrow \mathcal{D}_\mu J^\mu = 0$

$\partial_\mu J^\mu = ig[A_\mu, J^\mu] \Rightarrow$ if we start with current $J^{(1)}$ and field $A^{(1)}$ and want to calculate corrections $A^{(2)}$ and $J^{(2)}$ such that:

$$\begin{cases} A_\mu = A_\mu^{(1)} + A_\mu^{(2)} + \dots \\ J_\mu = J_\mu^{(1)} + J_\mu^{(2)} + \dots \end{cases}$$

$\Rightarrow$ then $\partial_\mu J^{(2)\mu} = ig[A_\mu^{(1)}, J^{(1)\mu}]$

and $[\mathcal{D}_{\mu\nu}^{ab}]^{-1} A_\nu^{(2)b} = g f^{abc} A_\nu^{(1)b} \partial_\mu A_\nu^{(1)c} + \dots + \overbrace{\frac{g^2}{4} A^4}^{\text{small}} \, J_\mu^{(2)a}$

Finally,

$$\boxed{\underline{A}(\underline{x}, x_-) = \frac{g}{2\pi} \sum_{a=1}^{N_c^2-1} \sum_{i=1}^{A} (T_i^a) \left\{ S(\underline{x}, x_{i-}) T^a S^{-1}(\underline{x}, x_{i-}) \frac{\underline{x}-\underline{x}_i}{|\underline{x}-\underline{x}_i|^2} \cdot \theta(x_- - x_{i-}) - S(\underline{x}, x'_{i-}) T^a S^{-1}(\underline{x}, x'_{i-}) \frac{\underline{x}-\underline{x}'_i}{|\underline{x}-\underline{x}'_i|^2} \theta(x_- - x'_{i-}) \right\}}$$

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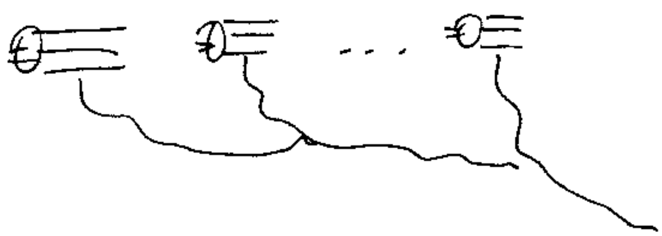

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To get equations of motion require that $\frac{\delta \mathcal{L}}{\delta A^a_\mu} = 0$

One gets

$$[D^{ab}_{\mu\nu}]^{-1} A^b_\nu = g f^{abc} A^b_\nu \partial_\mu A^c_\nu + \ldots + \frac{g^2}{4} f^{abc} f^{cde} A^b_\nu A^c_\mu A^d_\nu = J^a_\mu$$

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if A is small, we neglect all higher powers of A:

$$[D^{ab}_{\mu\nu}]^{-1} A^{(1)b}_\nu = J^{(1)a}_\mu \Rightarrow A^{(1)a}_\mu = -[D^{ab}_{\mu\nu}] \cdot J^{(1)b}_\nu$$

↖ propagator

To go to next order need to impose current conservation : $D_\mu F^{\mu\nu} = J^\nu \Rightarrow D_\mu J^\mu = 0$

$\partial_\mu J^\mu = ig[A_\mu, J^\mu] \Rightarrow$ if we start with current $J^{(1)}$ and field $A^{(1)}$ and want to calculate corrections $A^{(2)}$ and $J^{(2)}$ such that:

$$\begin{cases} A_\mu = A_\mu^{(1)} + A_\mu^{(2)} + \ldots \\ J_\mu = J_\mu^{(1)} + J_\mu^{(2)} + \ldots \end{cases}$$

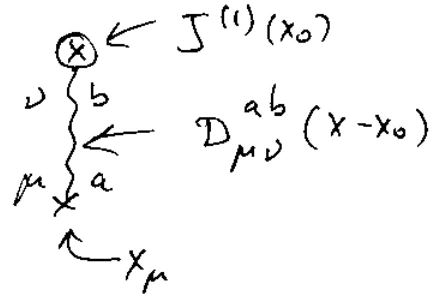
$$\begin{cases} \Rightarrow \text{then } \partial_\mu J^{(2)\mu} = ig[A^{(1)}_\mu, J^{(1)\mu}] \\ \text{and } [D^{ab}_{\mu\nu}]^{-1} A^{(2)b}_\nu = g f^{abc} A^{(1)b}_\nu \partial_\mu A^{(1)c}_\nu + \ldots + \frac{g^2}{4} \overbrace{A^4}^{\text{small}} = J^{(2)a}_\mu \end{cases}$$

Therefore $J^{(2)\mu} = ig \frac{1}{\partial_\mu}\left[A_\nu^{(1)}, J^{(1)\nu} \right]$ and

$$\left[D_{\mu\nu}^{ab} \right]^{-1} A_\nu^{(2)b} = g f^{abc} A_\nu^{(1)b} \partial_\mu A_\nu^{(1)c} + \dots + ig f^{abc} \frac{1}{\partial_\mu} A_\nu^{(1)b} J^{(1)\nu c}$$

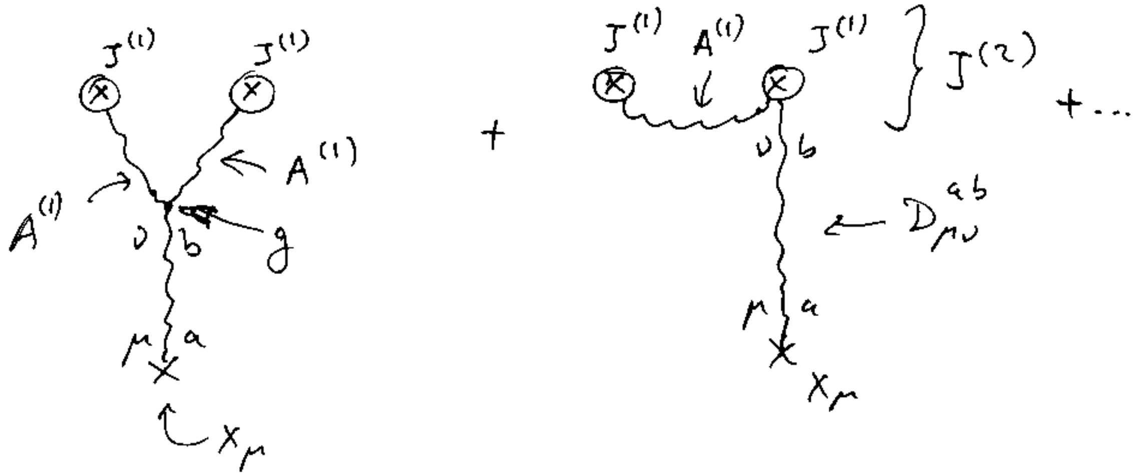
Diagrammatic interpretation: denote current by ⊗.

$A^{(1)a}_\mu (x) =$



lowest order field.

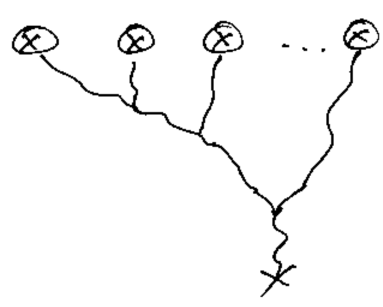
$A^{(2)a}_\mu (x) =$



$\Rightarrow$ One always has $\geqslant$ **1** fields on the right hand side $\Rightarrow$ the number of fields decreases (increases) as we go down (up) the diagram

$\Rightarrow$ Classical fields $\Leftrightarrow$ tree diagrams

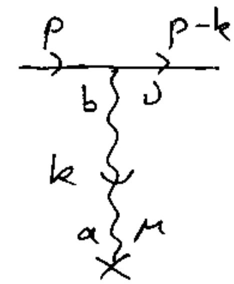
(to get classical physics from Q.F.T. ~ drop the loops)



~ all orders

Does this work in our case?

(I) Lowest order field: source current is a point charge (quark).



In covariant gauge ($\partial_\mu A^\mu = 0$)

$$A_\mu^{(1)\,a} = -\underbrace{D_{\mu\nu}^{ab}}_{\frac{-i g_{\mu\nu}\delta^{ab}}{k^2}} J_\nu^{(1)\,b} = \frac{+i g_{\mu\nu}\delta^{ab}}{k^2}\cdot\frac{i g T^b \bar{u}(p-k)\cdot\gamma_\nu u(p)}{2p_+} =$$

as $(p-k)^2 \approx -2p_+k_- = 0 \Rightarrow k_- = 0$

$$= -\frac{g}{k^2}T^a \underbrace{\bar{u}(p-k)\gamma_\mu u(p)}_{\delta_{\mu+}\,2p_+}\frac{1}{2p_+} = +\frac{g}{\underline{k}^2}\delta_{\mu+}T^a$$

Let's Fourier-transform it into coordinate space:

$$A_\mu^a(x) = \int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\cdot A_\mu^a(k)\cdot 2\pi\delta((p-k)^2) = \int\frac{d^2k\,dk_+}{(2\pi)^3\underbrace{2p_+}_{\text{already included}}}\cdot$$

$$\cdot e^{-ik_+x_- + i\underline{k}\cdot\underline{x}}\cdot A_\mu^a(k) = +g\,\delta_{\mu+}T^a\cdot\int\frac{d^2k\,dk_+}{(2\pi)^3}\cdot$$

$$\cdot e^{-ik_+x_- + i\underline{k}\cdot\underline{x}}\frac{1}{\underline{k}^2} = +g\,\delta_{\mu+}T^a\delta(x_-)\cdot\underbrace{\int\frac{d^2k}{(2\pi)^2}e^{+i\underline{k}\cdot\underline{x}}\frac{1}{\underline{k}^2}}_{=-\frac{1}{2\pi}\ln(|\underline{x}|\Lambda)} =$$

$$= -\frac{g}{2\pi}\delta_{\mu+}T^a\delta(x_-)\ln(|\underline{x}|\Lambda).$$

(math formula $\approx\frac{1}{(2\pi)^2}\int_\Lambda^{1/x}\frac{dk\cdot k\cdot d\varphi}{k^2} = \frac{1}{2\pi}\ln\frac{1}{x\Lambda}$)

(exactly as we had before!)

that's all there is in covariant gauge for large nucleus!

Last time we found the gluon field in $A^+ = 0$ light-cone gauge to be

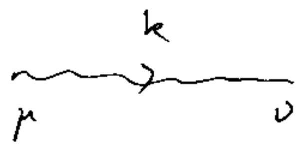
$$A^a_\mu(k) = g\, T^a \frac{k_\mu^\perp}{\underline{k}^2} \frac{1}{k_+}$$

Fourier transforming it we get

use $\int \frac{d^2 k}{(2\pi)^2} e^{i\underline{k}\cdot\underline{x}} \frac{\underline{k}}{\underline{k}^2} = \frac{i}{2\pi} \frac{\underline{x}}{\underline{x}^2}$

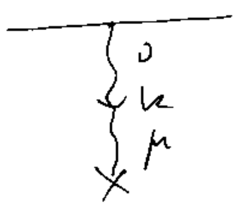
$$A^a_\mu(x) = \int \frac{d^2k\, dk_+}{(2\pi)^3} e^{-ik_+ x_- + i\underline{k}\cdot\underline{x}}\, g\, T^a \frac{k_\mu^\perp}{\underline{k}^2}\frac{1}{k_+} =$$

$$= g\, T^a \frac{+i}{2\pi}\frac{x_\mu^\perp}{\underline{x}^2} \cdot \int\limits_{-\infty}^{\infty} \frac{dk_+}{2\pi}\underbrace{\frac{1}{k_+}}_{\text{need to regulate!}} e^{-ik_+ x_-}$$



$$D_{\mu\nu}(k) = \frac{-i}{k^2+i\varepsilon}\Big[g_{\mu\nu} - \underbrace{\overset{\text{flows to } k_\nu}{\frac{\eta_\mu k_\nu}{k_+ + i\varepsilon}} - \overset{\text{flows from } k_\mu}{\frac{\eta_\nu k_\mu}{k_+ - i\varepsilon}}}\Big]$$

$\Rightarrow$ fixing <u>residual</u> gauge freedom: let's regulate this way



$$\sim \underbrace{\frac{\eta_\nu k_\mu}{k_+ + i\varepsilon}}_{\text{left with this.}} + \frac{\eta_\mu \cancel{k_\nu}^{\,0}}{k_+ - i\varepsilon} \quad (\text{remember } \bar{u}(p-k)\,\not{k}\, u(p) = 0)$$

$$\Rightarrow A^a_\mu(x) = g\, T^a \frac{i}{2\pi}\frac{x_\mu^\perp}{\underline{x}^2} \underbrace{\int\limits_{-\infty}^{\infty} \frac{dk_+}{2\pi}\frac{1}{k_+ + i\varepsilon} e^{-ik_+ x_-}}_{\frac{1}{2\pi}(-2\pi i)\,\theta(x_-)} = \frac{g}{2\pi} T^a\, \theta(x_-) \frac{x_\mu^\perp}{|\underline{x}|^2}$$

(Compare $\underline{A} = \frac{g}{2\pi}\sum_a \sum_i (T^a_i)\Big\{ \underset{\substack{\shortparallel \\ (\text{L.O.})}}{S(\underline{x}, x_{i-})}\, T^a\, \underset{\shortparallel}{S^{-1}} \frac{\underline{x}-\underline{x}_i}{|\underline{x}-\underline{x}_i|^2}\,\theta(x_- - x_{i-}) - \ldots \Big\}$

$= \frac{g}{2\pi}\sum_a \sum_i (T^a_i)\, T^a \Big\{ \theta(x_- - x_{i-}) \frac{\underline{x}-\underline{x}_i}{|\underline{x}-\underline{x}_i|^2} - \ldots \Big\}$ the same!)

II Next-to-leading order field: ($\neq 0$ in L (gauge))

Take a nucleus made out of 2 nucleons ($A = 2$)

$$S(\underline{x}, x_-) = \exp\left[\frac{ig^2}{2\pi} \sum_a T^a (T^a_1)\, \theta(x_- - x_{-1}) \ln\left(\frac{|\underline{x} - \underline{x}_1|}{|\underline{x} - \underline{x}_1'|} \right) \right]$$

for $x_{1-} < x_{2-}$:

Forgetting about antiquarks write

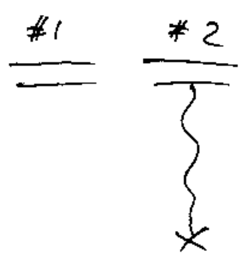
$$\underline{A}(\underline{x}, x_-) = \frac{g}{2\pi} \sum_a \sum_{i=1}^{2} (T^a_i) \left\{ S(\underline{x}, x_{i-})\, T^a S^{-1}(\underline{x}, x_{i-}) \frac{\underline{x} - \underline{x}_i}{|\underline{x} - \underline{x}_i|^2} \theta(x_- - x_{i-}) - \ldots \right\}$$

$$= \frac{g}{2\pi} \sum_a (T^a_1)\, T^a \left\{ \frac{\underline{x} - \underline{x}_1}{|\underline{x} - \underline{x}_1|^2} \theta(x_- - x_{1-}) - \ldots \right\} +$$

// field of nucleon #1

$$+ \frac{g}{2\pi} \sum_a (T^a_2)\, T^a \left\{ \frac{\underline{x} - \underline{x}_2}{|\underline{x} - \underline{x}_2|^2} \theta(x_- - x_{2-}) - \ldots \right\}$$

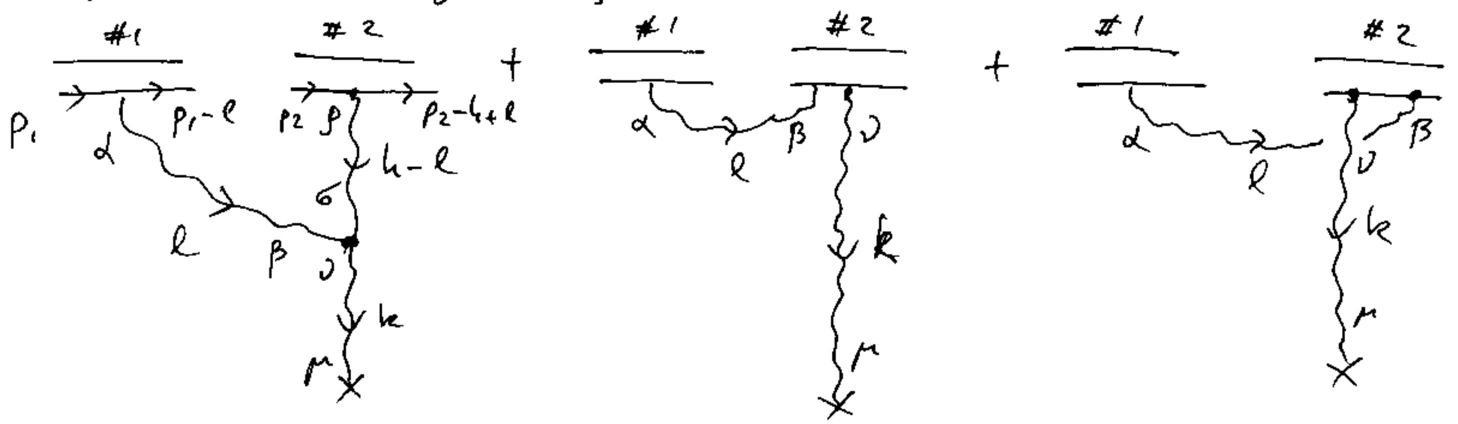
//



field of nucleon #2

$$+ \frac{g^3}{(2\pi)^2} T^a f^{abc} \cdot (T_2^b)(T_1^c) \frac{\underline{x}-\underline{x}_2}{|\underline{x}-\underline{x}_2|^2} \ln\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|}\right) \theta(x_- - x_{2-}) - \ldots$$

relevant diagrams



$$\int \frac{d^4\ell}{(2\pi)^4} e^{-i\ell\cdot(x_2-x_1)} 2\pi\,\delta((p_1-\ell)^2) = \frac{1}{2p_{1+}} \int \frac{d^2\ell\, d\ell_+}{(2\pi)^3} e^{-i\ell_+\Delta x_- + i\underline{\ell}\cdot(\underline{x}_2-\underline{x}_1)}$$

$$\Delta x_- = x_{2-} - x_{1-} > 0$$

$\Rightarrow$ look at the propagator's Fourier transform:

$$\int \frac{d\ell_+}{2\pi} e^{-i\ell_+\Delta x_-} \frac{i}{|\underline{\ell}|^2}\left[g_{\alpha\beta} - \frac{\eta_\alpha \ell_\beta}{\ell_+ + i\varepsilon} - \frac{\eta_\beta \cancel{\ell_\alpha}^{\,0}}{\ell_+ - i\varepsilon}\right]\left[g_{\rho\sigma} - \frac{\eta_\rho (k-\ell)_\sigma}{k_+ - \ell_+ + i\varepsilon} - \right.$$

$$\left. - \frac{\eta_\sigma \cancel{(k-\ell)_\rho}^{\,0}}{k_+ - \ell_+ - i\varepsilon}\right] \frac{i}{|\underline{k}-\underline{\ell}|^2} \Gamma_{\beta\nu\sigma}(k,\ell)\, \bar{u}(p_1-\ell)\gamma_\alpha u(p_1)\, \bar{u}(p_2-k+\ell)\gamma_\rho \cdot$$

$\cdot\, u(p_2)$

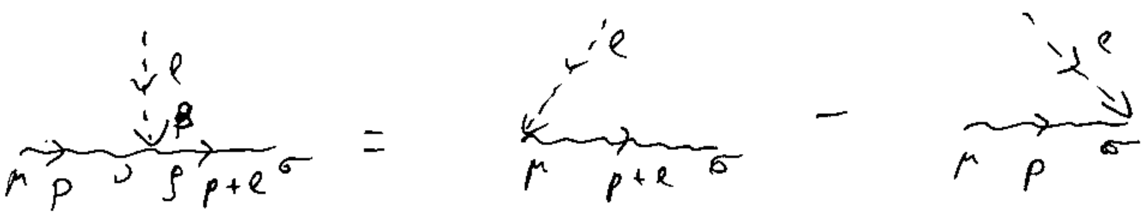
follow $g_{\alpha\beta}$: $g_{\alpha\beta}\, g_{\rho\sigma} \Rightarrow$ no ℓ_+ denominator $\Rightarrow \propto \delta(\Delta x_-) = 0$.

$$g_{\alpha\beta} \frac{\eta_\rho (k-\ell)_\sigma}{k_+ - \ell_+ + i\varepsilon} \Rightarrow \sim \theta(-\Delta x_-) = 0 \quad \text{as } \Delta x_- > 0.$$

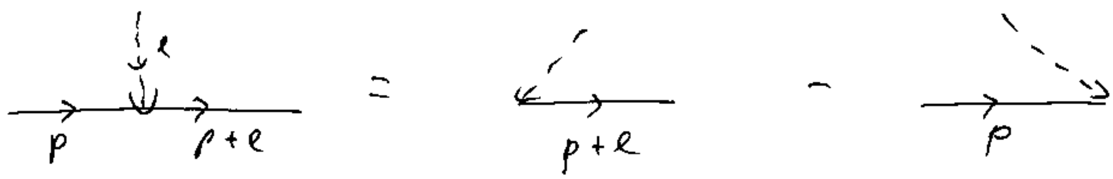
$\Rightarrow$ only $\dfrac{\eta_\alpha l_\beta}{l_+ + i\varepsilon}$ contributes

$\Rightarrow$ one can show that it's true for the other two graphs (much easier)

$\Rightarrow$ Use Ward identities : $\overset{l}{\dashrightarrow}{}^{\beta}$ is a gluon line with l_β (longitudinally polarized)

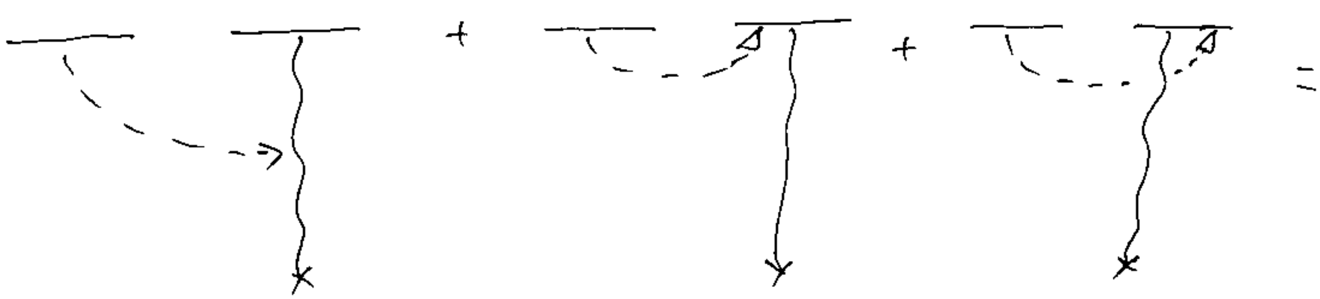


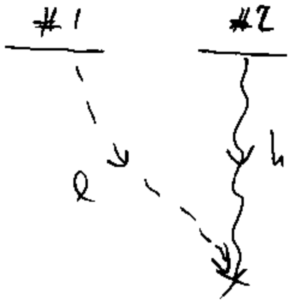
$$D_{\mu\nu}(p)\, l_\beta\, \Gamma_{\beta\nu\rho}\, D_{\rho\sigma}(p+l) = (-i)\, D_{\mu\sigma}(p+l) - (-i)\, D_{\mu\sigma}(p)$$



$$\frac{i}{\not p}\,\not l\,\frac{i}{\not p+\not l} = (-i)\left[\frac{i}{\not p+\not l} - \frac{i}{\not p}\right]$$

Applying Ward identities one gets:





$$\simeq (-i) g^3 \frac{1}{\underline{l}^2} \frac{1}{l_+ + i\varepsilon} \cdot \frac{(k-l)^\perp_\mu}{(\underline{k}-\underline{l})^2} \frac{1}{k_+ - l_+ + i\varepsilon} \underbrace{f^{abc} (T_2^b)(T_1^c)}_{\text{can check}}$$

Fourier-transforming:

$$A_\mu^a(x) \simeq -i g^3 f^{abc} (T_2^b)(T_1^c) \int \frac{d^2k\, dk_+}{(2\pi)^3} e^{-ik_+(x_- - x_{2-}) + i\underline{k}\cdot(\underline{x}-\underline{x}_2)} \cdot$$

$$\cdot \frac{d^2l\, dl_+}{(2\pi)^3} e^{-il_+(x_{2-} - x_{1-}) + i\underline{l}\cdot(\underline{x}_2 - \underline{x}_1)} \frac{(k-l)^\perp_\mu}{\underline{l}^2(\underline{k}-\underline{l})^2} \cdot \frac{1}{(l_+ + i\varepsilon)(k_+ - l_+ + i\varepsilon)} =$$

$$= +i g^3 f^{abc} (T_2^b)(T_1^c) \cdot \int \frac{d^2k\, d^2l}{(2\pi)^4} e^{i\underline{k}\cdot(\underline{x}-\underline{x}_2) + i\underline{l}\cdot(\underline{x}_2 - \underline{x}_1)} \cdot$$

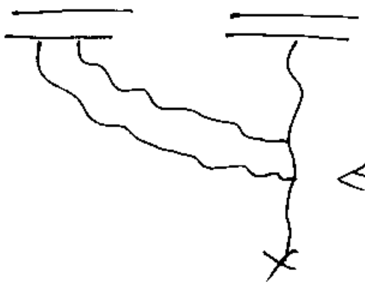
$$\cdot \frac{(k-l)^\perp_\mu}{\underline{l}^2(\underline{k}-\underline{l})^2} \theta(x_- - x_{2-}) = +i g^3 f^{abc} (T_2^b)(T_1^c)\, \theta(x_- - x_{2-}) \cdot$$

$$\cdot \frac{-i}{(2\pi)^2} \frac{\underline{x} - \underline{x}_2}{|\underline{x} - \underline{x}_2|^2} \cdot \ln(|\underline{x} - \underline{x}_1|\Lambda) = \frac{g^3}{(2\pi)^2} f^{abc} (T_2^b)(T_1^c)\, \theta(x_- - x_{2-}) \cdot$$

$$\cdot \frac{\underline{x} - \underline{x}_2}{|\underline{x} - \underline{x}_2|^2} \ln(|\underline{x} - \underline{x}_1|\Lambda)$$ as desired!

$\Rightarrow$ we established correspondence between classical fields and tree-level diagrams!

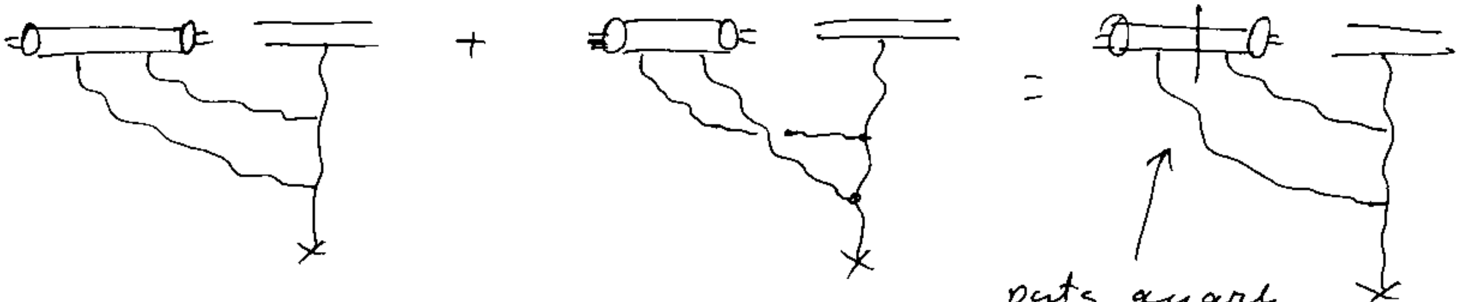
Limits of applicability: expand S to $o(g^4)$:



$+ \ \ldots$

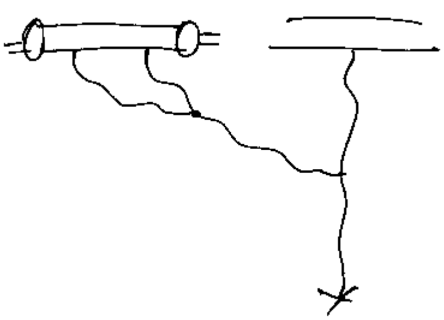
$\leftarrow$ appears to be a loop ?!

$\Rightarrow$ require color-neutrality and add crossed graph:



puts quark on mass shell

$\Rightarrow$ now it's like 2 independent rescatterings $\sim$ still classical

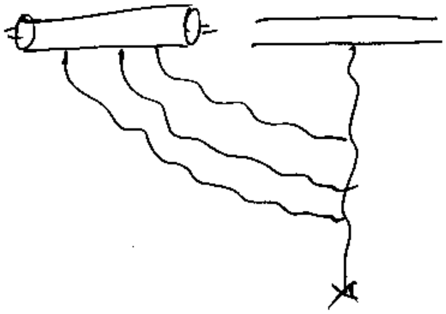


$= 0$ (color neutrality)

a quantum graph

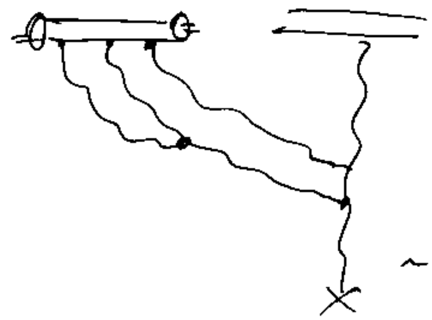
Quantum (running coupling) loops cancel at this order!

Next order:



~ classical graph

~ $O(g^7)$



$\neq 0$

~ $O(g^7)$

quantum graph with a loop

$\Rightarrow$ at the level of 3 extra gluons quantum loops don't cancel anymore!

$\Rightarrow$ classical fields dominate at the order of no more than <u>2 gluons per nucleon</u>!

2 gluons ~ $g^4 \sim \alpha_s^2$

nucleons at given impact ~ $A^{1/3}$ parameter

$\boxed{\alpha_s^2 A^{1/3} \sim 1}$ is a new resummation parameter

$\alpha_s \ll 1 \qquad A^{1/3} \gg 1 \qquad \Rightarrow \alpha_s^2 A^{1/3} \sim 1$ (like leading logs)

$\Rightarrow$ Classical fields resum powers of $\alpha^2 A^{1/3}$, quantum loops bring in $O(\alpha_s)$ corrections.

Now we'll use the WW field $\underline{A}(x)$ that we found to calculate unintegrated gluon distribution

$$\varphi(x, k_T^2) \equiv \frac{k_+^2}{(2\pi)^2} \left\langle \underline{A}^a(-k) \cdot \underline{A}^a(k) \right\rangle$$

We found the field in coordinate space $\Rightarrow$ have to transform

$$A_\mu^a(k) = \int d^2x \, dx_- \, e^{ik_+x_- - i\underline{k}\cdot\underline{x}} A_\mu^a(x)$$

such that

$$\varphi(x, k_T^2) = \frac{k_+^2}{(2\pi)^2} \int d^2x \, dx_- \, d^2y \, dy_- \, e^{-ik_+x_- + i\underline{k}\cdot\underline{x} + ik_+y_- - i\underline{k}\cdot\underline{y}} \cdot$$

$$\cdot \left\langle \underline{A}^a(x) \cdot \underline{A}^a(y) \right\rangle = \frac{1}{(2\pi)^2} \int d^2x \, d^2y \, dx_- dy_- \, e^{-ik_+(x_- - y_-) + i\underline{k}\cdot(\underline{x} - \underline{y})} \cdot$$

$$\cdot \left\langle \underbrace{\partial_+ \underline{A}^a(x)}_{F_{+i}^a(x)} \cdot \underbrace{\partial_+ \underline{A}^a(y)}_{F_{+i}^a(y)} \right\rangle = \frac{1}{(2\pi)^2} \int d^2x \, d^2y \, dx_- dy_- \, e^{ik_+(y_- - x_-) - i\underline{k}\cdot(\underline{y} - \underline{x})} \cdot$$

$$\cdot \left\langle F_{+i}^a(x) \, F_{+i}^a(y) \right\rangle .$$

We have to plug in

$$F_{+i}(x) = \frac{g}{2\pi} \sum_{a=1}^{N_c^2-1} \sum_{j=1}^{A} \left(T_j^a\right) \left\{ S(\underline{x}, x_{j-}) \, T^a \, S^{-1}(\underline{x}, x_{j-}) \frac{(\underline{x} - \underline{x}_j)_i}{|\underline{x} - \underline{x}_j|^2} \cdot \right.$$

$$\left. \cdot \, \delta(x_- - x_{j-}) \; - \; \text{anti-quarks} \right\} .$$

$\Rightarrow$ to average $\langle \ldots \rangle$ need a model for nuclear wave function.

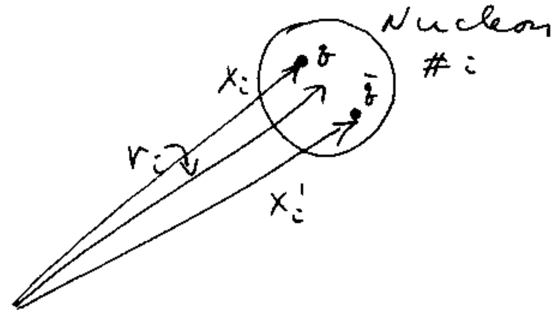
Assume that $q\bar{q}$ are equally probable to be anywhere in nucleons + nucleons can be anywhere in the nucleus with equal probability.

Then:

trace in color space of ith nucleon

$$\langle \ldots \rangle = \prod_{i=1}^{A} \int \frac{d^3 r_i}{V_A} \cdot \frac{d^3 x_i \, d^3 x_i'}{V_N^2} \frac{1}{N_c} Tr_i [\ldots]$$

where V_A & V_N are volumes of the nucleus and of a nucleon



$$\frac{d^3 x_i}{V_N} = \frac{d^2 x_i \, d x_{i-}}{S_N \cdot \Delta a_-} \quad , \quad \frac{d^3 r_i}{V_A} = \frac{d^2 r_i \, d r_{i-}}{S_N \cdot \frac{R}{\gamma}\sqrt{2}}$$

Plugging fields in: $\left(F_{+i}^a F_{+i}^a = 2 Tr [F_{+i} F_{+i}] \right)$

$$\varphi(x, k_T^2) = 2 \int \frac{d^2 x \, d^2 y}{(2\pi)^2} dx_- dy_- \, e^{i k_+ (y_- - x_-) - i \underline{k} \cdot (\underline{y} - \underline{x})} \prod_{i=1}^{A} \int \frac{d^3 r_i}{V_A} \frac{d^3 x_i \, d^3 x_i'}{V_N^2} \cdot$$

$$Tr \left\{ \frac{Tr_i}{N_c} \left[\frac{g^2}{(2\pi)^2} \sum_{j,k=1}^{A} (T_j^a)(T_k^b) \, S(\underline{x}, x_{j-}) T^a S^{-1}(\underline{x}, x_{j-}) \cdot \right. \right.$$

$$\cdot S(\underline{y}, x_{k-})\, T^b\, S^{-1}(\underline{y}, x_{k-}) \Big] \Big\} \delta(x_- - x_{j-})\, \delta(y_- - x_{k-}) \cdot$$

$$\cdot \frac{\underline{x} - \underline{x}_j}{|\underline{x} - \underline{x}_j|^2} \cdot \frac{\underline{y} - \underline{x}_k}{|\underline{y} - \underline{x}_k|^2} - \left(\text{anti-quarks} + \ldots\right)$$

Color traces demand that $j = k$ (there is no (T_j^a) is $S(\underline{x}, x_{j-})$, no (T_k^b) in $S(\underline{y}, x_{k-})$) such that, as $\frac{Tr_j}{N_c}(T_j^a)(T_j^b) = \frac{1}{2N_c}\delta^{ab}$

$$\varphi(x, k_T^2) = \frac{g^2}{N_c (2\pi)^2} \int \frac{d^2x\, d^2y}{(2\pi)^2}\, e^{-i\underline{k}\cdot(\underline{y} - \underline{x})} \prod_{i=1}^{j} \int \frac{d^3 r_i}{V_A} \frac{d^3 x_i\, d^3 x_i'}{V_N^2} \cdot$$

$$Tr \Big\{ \sum_{j=1}^{A} \frac{Tr_i}{N_c} \Big[S(\underline{x}, x_{j-})\, T^a\, S^{-1}(\underline{x}, x_{j-})\, S(\underline{y}, x_{j-})\, T^a\, S^{-1}(\underline{y}, x_{j-}) \Big] \Big\} \cdot$$

$$\cdot \frac{\underline{x} - \underline{x}_j}{|\underline{x} - \underline{x}_j|^2} \cdot \frac{\underline{y} - \underline{x}_j}{|\underline{y} - \underline{x}_j|^2} - \left(\text{anti-quarks} + \ldots\right)$$

In each term of the sum over j, the jth nucleon drops out of S-matrices $\Rightarrow$ can average over $\underline{x}_j$ (also, there's no x_j' or r_j)

$$\int \frac{d^2 x_j}{S_A} \frac{\underline{x} - \underline{x}_j}{|\underline{x} - \underline{x}_j|^2} \cdot \frac{\underline{y} - \underline{x}_j}{|\underline{y} - \underline{x}_j|^2} = \int \frac{d^2 x_j}{S_A} \int \frac{d^2 \ell\, d^2 q}{(2\pi)^2}\, e^{i\underline{\ell}\cdot(\underline{x} - \underline{x}_j) - i\underline{q}\cdot(\underline{y} - \underline{x}_j)} \cdot$$

$$\cdot \frac{\underline{\ell}}{\underline{\ell}^2} \cdot \frac{\underline{q}}{\underline{q}^2} = \frac{1}{S_A} \int \frac{d^2 \ell}{\underline{\ell}^2}\, e^{i\underline{\ell}\cdot(\underline{x} - \underline{y})} = \frac{1}{S_A} \cdot 2\pi \cdot \ln \frac{1}{|\underline{x} - \underline{y}|\Lambda}$$

Such that

$$\varphi(x, k_T^2) = \frac{g^2}{(2\pi)^2 N_c}\int \frac{d^2x\, d^2y}{(2\pi)^2} e^{-i\underline{k}\cdot(\underline{y}-\underline{x})} \frac{2\pi}{S_A} \ln\left(\frac{1}{|\underline{x}-\underline{y}|\Lambda}\right) \cdot \sum_{j=1}^{A}$$

$$\cdot \prod_{i=1}^{j-1} \int \frac{d^3 r_i}{V_A} \frac{d^3 x_i\, d^3 x_i'}{V_N^2} \frac{Tr_i}{N_c} \left\{ Tr\left[S(\underline{x}, x_{j-}) T^a S^{-1}(\underline{x}, x_{j-}) S(\underline{y}, x_{j-}) \cdot \right.\right.$$

$$\left.\left. \cdot T^a S^{-1}(\underline{y}, x_{j-}) \right] \right\} - (\text{anti-quarks})$$

$\hookrightarrow$ just modify $\ln \frac{1}{|\underline{x}-\underline{y}|\Lambda} \to 2 \ln \frac{a}{|\underline{x}-\underline{y}|}$ (nucleon radius $\to a$)

Now,

$$S(\underline{x}, x_{j-}) = \prod_{i=1}^{A} \exp\left\{ \frac{i g^2}{2\pi} T^a (T_i^a) \ln\left(\frac{|\underline{x}-\underline{x}_i|}{|\underline{x}-\underline{x}_i'|}\right) \theta(x_{j-} - x_{i-}) \right\} =$$

$$= \prod_{i=1}^{j-1} \exp\left\{ \frac{i g^2}{2\pi} T^a (T_i^a) \ln\left(\frac{|\underline{x}-\underline{x}_i|}{|\underline{x}-\underline{x}_i'|}\right) \right\} =$$

$$\cdot \left[1 + \frac{i g^2}{2\pi} T^a (T_1^a) \ln\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|}\right) - \frac{g^4}{2(2\pi)^2} T^a T^b (T_1^a)(T_1^b) \ln^2\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1|}\right) \right] \cdot$$

no more, 2 gluons per nucleon.

$$\cdot \prod_{i=2}^{j-1} \exp\{\ldots\}$$

$$\Rightarrow S^{-1}(\underline{x}, x_{j-})\, S(\underline{y}, x_{j-}) = \prod_{i=j-1}^{i=2} \exp\{\ldots\} \cdot \left[1 \blacksquare \frac{i g^2}{2\pi} T^a (T_1^a) \cdot \right.$$

$$\left. \cdot \ln\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|} \cdot \frac{|\underline{y}-\underline{x}_1'|}{|\underline{y}-\underline{x}_1|}\right) - \frac{g^4}{2(2\pi)^2} T^a T^b (T_1^a)(T_1^b) \left(\ln^2\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|} \cdot \frac{|\underline{y}-\underline{x}_1'|}{|\underline{y}-\underline{x}_1|}\right)\right) \right] \cdot$$

$$\cdot \prod_{k=2}^{j-1} \exp\{\ldots\}$$

the other combination $S^{-1}(\underline{y}, x_{j-}) S'(\underline{x}, x_{j-})$ can be rewritten similarly (with $\underline{x} \leftrightarrow \underline{y}$).

Taking color traces we write $\left(\mathrm{Tr}(T_1^a T_1^b) = \frac{1}{2}\delta^{ab}, \quad T^a T^b \frac{1}{2}\delta^{ab} = \frac{C_F}{2}\right)$

$\mathrm{Tr}[T^a X T^a Y] = -\frac{1}{2N_c}\mathrm{Tr}\,XY$ if $\mathrm{Tr}\,X = \mathrm{Tr}\,Y = 0$.

$$\mathrm{Tr}\left[S'(\underline{x}, x_{j-})\, T^a S^{-1}(\underline{x}, x_{j-})\, S(\underline{y}, x_{j-})\, T^a S^{-1}(\underline{y}, x_{j-})\right] =$$

$$= \left[1 - \frac{g^4}{(2\pi)^2}\,\frac{1}{4}\,\ln^2\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|}\cdot\frac{|\underline{y}-\underline{x}_1'|}{|\underline{y}-\underline{x}_1|}\right)\right]\cdot \mathrm{Tr}\left[S'(\underline{x}, x_{j-})\, T^a S'^{-1}(\underline{x}, x_{j-})\cdot\right.$$

$$\left.\cdot S'(\underline{y}, x_{j-})\, T^a S'^{-1}(\underline{y}, x_{j-})\right]$$ where prime denotes that we dropped the 1st nucleon.

Averaging

$$\int \frac{d^2x_1}{S_A} \ln(|\underline{x}-\underline{x}_1|\Lambda)\,\ln(|\underline{y}-\underline{x}_1|\Lambda) = \int\frac{d^2x_1}{S_A}\cdot\int\frac{d^2l\,d^2q}{(2\pi)^2}\,\frac{e^{i\underline{l}\cdot(\underline{x}-\underline{x}_1) - i\underline{q}\cdot(\underline{y}-\underline{x}_1)}}{l^2 q^2} =$$

$$= \frac{1}{S_A}\int\frac{d^2l}{l^4}\,e^{i\underline{l}\cdot(\underline{x}-\underline{y})} = -\frac{1}{S_A}\int\frac{d^2l}{l^4}\left[1 - e^{i\underline{l}\cdot(\underline{x}-\underline{y})}\right] =$$

$$= -\frac{1}{S_A}\,2\pi\int_0^\infty \frac{dl}{l^3}\left(1 - J_0(l|\underline{x}-\underline{y}|)\right) = -\frac{2\pi}{S_A}\,\frac{1}{4}|\underline{x}-\underline{y}|^2\cdot\ln\frac{1}{|\underline{x}-\underline{y}|\Lambda}$$

$\Rightarrow$ assuming that $|\underline{x}-\underline{y}| \ll a$ (nucleon's radius)

$$[1 - \ldots] = 1 - \frac{g^4}{(2\pi)^2}\,\frac{1}{2}\,\frac{2\pi}{4S_A}\,|\underline{x}-\underline{y}|^2\cdot\ln\frac{|\underline{x}-\underline{y}|^2}{a^2}.$$

Iterating the procedure for all the nucleons we get

$$C_F N_c \left[1 - \frac{\overset{=(4\pi\alpha)^2}{g^4}}{16\,\pi S_A} |\underline{x}-\underline{y}|^2 \ln \frac{|\underline{x}-\underline{y}|^2}{a^{-2}} \right]^{j-1} \simeq$$

$$\simeq C_F N_c\, e^{-\frac{\pi\alpha^2}{S_A} |\underline{x}-\underline{y}|^2 \ln \frac{|\underline{x}-\underline{y}|^2}{a^{-2}} \cdot (j-1)}$$

Such that

$$\varphi(x, k_T^2) = \frac{g^2}{2\pi S_A N_c} \cdot \ln \frac{a^2}{|\underline{x}-\underline{y}|^2} \cdot \int \frac{d^2x\, d^2y}{(2\pi)^2} e^{-i\underline{k}\cdot(\underline{y}-\underline{x})} \cdot C_F N_c$$

$$\cdot \underbrace{\sum_{j=1}^{A} e^{-\frac{\pi\alpha^2}{S_A}(j-1)|\underline{x}-\underline{y}|^2 \ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}}}.$$

Rewrite as $\displaystyle \frac{A}{2\sqrt{R^2-b^2}} \int_0^{2\sqrt{R^2-b^2}} dz \cdot e^{-\frac{\pi\alpha^2}{V_A} zA|\underline{x}-\underline{y}|^2 \ln \frac{|\underline{x}-\underline{y}|^2}{a^{-2}}} =$

$$= + \frac{V_A}{\pi\alpha^2 \cancel{A}} \frac{1}{|\underline{x}-\underline{y}|^2 \ln \frac{|\underline{x}-\underline{y}|^2}{a^{-2}}} \left[1 - e^{-\frac{\pi\alpha^2}{S_A}\cdot A |\underline{x}-\underline{y}|^2 \ln \frac{|\underline{x}-\underline{y}|^2}{a^{-2}}} \right] \cdot$$

$$\cdot \frac{\cancel{A}}{2\sqrt{R^2-b^2}} = \frac{S_A}{\pi\alpha^2} \frac{1}{|\underline{x}-\underline{y}|^2 \ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}} \left[1 - e^{-\frac{\pi\alpha^2}{S_A} A |\underline{x}-\underline{y}|^2 \ln \frac{|\underline{x}-\underline{y}|^2}{a^{-2}}} \right].$$

giving

$$\varphi(x, k_T^2) = \int \frac{\cancel{2\alpha}}{S_A \cancel{N_c}} \cancel{\ln \frac{a^2}{|\underline{x}-\underline{y}|^2}} \cdot \frac{\cancel{S_A}}{\cancel{2}\pi\alpha^2} \frac{C_F \cancel{N_c}}{|\underline{x}-\underline{y}|^2 \cancel{\ln \frac{|\underline{x}-\underline{y}|^2}{a^{-2}}}} \cdot \frac{d^2x\, d^2y}{(2\pi)^2} e^{-i\underline{k}\cdot(\underline{x}-\underline{y})}.$$

Such that

$$\varphi(x, k_T^2) = \frac{g^2}{(2\pi)^2 N_c} \int \frac{d^2x\, d^2y}{(2\pi)^2}\, e^{-i\underline{k}\cdot(\underline{y}-\underline{x})}\, \frac{2\pi}{S_A} \ln\left(\frac{1}{|\underline{x}-\underline{y}|\Lambda}\right) \cdot \sum_{j=1}^{A}$$

$$\cdot \prod_{i=1}^{j-1} \int \frac{d^3r_i}{V_A}\, \frac{d^3x_i\, d^3x_i'}{V_N^2}\, \frac{Tr_i}{N_c} \left\{ Tr\left[S(\underline{x}, x_{j-})\, T^a\, S^{-1}(\underline{x}, x_{j-})\, S(\underline{y}, x_{j-}) \cdot \right.\right.$$

$$\left.\left. \cdot\, T^a\, S^{-1}(\underline{y}, x_{j-}) \right] \right\} - (\text{anti-quarks})$$

$\to$ just modify $\ln \frac{1}{|\underline{x}-\underline{y}|\Lambda} \to 2 \ln \frac{a}{|\underline{x}-\underline{y}|}$ ← nucleon radius

Now,

$$S(\underline{x}, x_{j-}) = \prod_{i=1}^{A} \exp\left\{ \frac{ig^2}{2\pi} T^a (T^a_i) \ln\left(\frac{|\underline{x}-\underline{x}_i|}{|\underline{x}-\underline{x}_i'|}\right) \theta(x_{j-} - x_{i-}) \right\} =$$

$$= \prod_{i=1}^{j-1} \exp\left\{ \frac{ig^2}{2\pi} T^a (T^a_i) \ln\left(\frac{|\underline{x}-\underline{x}_i|}{|\underline{x}-\underline{x}_i'|}\right) \right\} =$$

$$\cdot \left[1 + \frac{ig^2}{2\pi} T^a (T^a_1) \ln\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|}\right) - \frac{g^4}{2(2\pi)^2} T^a T^b (T^a_1)(T^b_1) \ln^2\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|}\right) \right] \cdot$$

$$\cdot \prod_{i=2}^{j-1} \exp\{\ldots\}$$

no more, 2 gluons per nucleon.

$$\Rightarrow S^{-1}(\underline{x}, x_{j-})\, S(\underline{y}, x_{j-}) = \prod_{i=j-1}^{i=2} \exp\{\ldots\} \cdot \left[1 + \frac{ig^2}{2\pi} T^a (T^a_1) \cdot \right.$$

$$\left. \cdot \ln\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|} \cdot \frac{|\underline{y}-\underline{x}_1'|}{|\underline{y}-\underline{x}_1|} \right) - \frac{g^4}{2(2\pi)^2} T^a T^b (T^a_1)(T^b_1) \left(\ln^2\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|} \cdot \frac{|\underline{y}-\underline{x}_1'|}{|\underline{y}-\underline{x}_1|} \right) \right) \right] \cdot$$

$$\cdot \prod_{k=2}^{j-1} \exp\{\ldots\}$$

the other combination $S^{-1}(\underline{y}, x_{j-})\, S(\underline{x}, x_{j-})$ can be rewritten similarly (with $\underline{x} \leftrightarrow \underline{y}$).

Taking color traces we write $\left(\mathrm{Tr}(T_1^a T_1^b) = \frac{1}{2}\delta^{ab},\ T^a T^b \frac{1}{2}\delta^{ab} = \frac{C_F}{2}\right)$

$\mathrm{Tr}[T^a X T^a Y] = -\frac{1}{2N_c}\mathrm{Tr}XY$ if $\mathrm{Tr}X = \mathrm{Tr}Y = 0$.

$$\mathrm{Tr}\left[S(\underline{x}, x_{j-})\, T^a\, S^{-1}(\underline{x}, x_{j-})\, S(\underline{y}, x_{j-})\, T^a\, S^{-1}(\underline{y}, x_{j-}) \right] =$$

$$= \left[1 - \frac{g^4}{(2\pi)^2} \frac{1}{4} \ln^2\left(\frac{|\underline{x} - \underline{x}_1|}{|\underline{x} - \underline{x}_1'|} \cdot \frac{|\underline{y} - \underline{x}_1'|}{|\underline{y} - \underline{x}_1|} \right) \right] \cdot \mathrm{Tr}\left[S'(\underline{x}, x_{j-})\, T^a\, S'^{-1}(\underline{x}, x_{j-}) \cdot S'(\underline{y}, x_{j-})\, T^a\, S'^{-1}(\underline{y}, x_{j-}) \right]$$

where prime denotes that we dropped the 1st nucleon.

Averaging

$$\int \frac{d^2 x_1}{S_A} \ln(|\underline{x} - \underline{x}_1|\Lambda) \ln(|\underline{y} - \underline{x}_1|\Lambda) = \int \frac{d^2 x_1}{S_A} \cdot \int \frac{d^2 l\, d^2 q}{(2\pi)^2} \frac{e^{i\underline{l}\cdot(\underline{x} - \underline{x}_1) - i\underline{q}\cdot(\underline{y} - \underline{x}_1)}}{l^2 q^2} =$$

$$= \frac{1}{S_A} \int \frac{d^2 l}{l^4} e^{i\underline{l}\cdot(\underline{x} - \underline{y})} = -\frac{1}{S_A} \int \frac{d^2 l}{l^4} \left[1 - e^{i\underline{l}\cdot(\underline{x} - \underline{y})} \right] =$$

$$= -\frac{1}{S_A} 2\pi \int_0^\infty \frac{dl}{l^3} \left(1 - J_0(l|\underline{x} - \underline{y}|) \right) = -\frac{2\pi}{S_A} \frac{1}{4} |\underline{x} - \underline{y}|^2 \cdot \ln \frac{1}{|\underline{x} - \underline{y}|\Lambda}$$

$\Rightarrow$ assuming that $|\underline{x} - \underline{y}| \ll a$ (nucleon's radius)

$$[1 - \ldots] = 1 - \frac{g^4}{(2\pi)^2} \frac{1}{2} \frac{2\pi}{4 S_A} |\underline{x} - \underline{y}|^2 \cdot \ln \frac{|\underline{x} - \underline{y}|^2}{a^2}.$$

$$\int \frac{d^2x_1}{S_A}\frac{d^2x_1'}{S_N}\cdot \ln^2\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|}\cdot\frac{|\underline{y}-\underline{x}_1'|}{|\underline{y}-\underline{x}_1|}\right) = \int \frac{d^2x_1\, d^2x_2}{S_A\, S_N}\cdot \int \frac{d^2q\, d^2l}{(2\pi)^2 \underline{q}^2 \underline{l}^2}\cdot$$

$$\cdot\left(e^{i\underline{q}\cdot(\underline{x}-\underline{x}_1)} - e^{i\underline{q}\cdot(\underline{x}-\underline{x}_1')} + e^{i\underline{q}\cdot(\underline{y}-\underline{x}_1')} - e^{i\underline{q}\cdot(\underline{y}-\underline{x}_1)}\right)\cdot$$

$$\cdot\left(e^{-i\underline{l}\cdot(\underline{x}-\underline{x}_1)} - e^{-i\underline{l}\cdot(\underline{x}-\underline{x}_1')} + e^{-i\underline{l}\cdot(\underline{y}-\underline{x}_1')} - e^{-i\underline{l}\cdot(\underline{y}-\underline{x}_1)}\right) =$$

$$= \Big|\, \underline{x}_1' = \underline{x}_1 + \underline{a} \;=\; \frac{1}{S_A}\int \frac{d^2 l}{\underline{l}^4}\left(e^{i\underline{l}\cdot\underline{x}} - e^{i\underline{l}\cdot(\underline{x}-\underline{a})} + e^{i\underline{l}\cdot(\underline{y}-\underline{a})} -\right.$$

$$\left. - e^{i\underline{l}\cdot\underline{y}}\right)\left(e^{-i\underline{l}\cdot\underline{x}} - e^{-i\underline{l}\cdot(\underline{x}-\underline{a})} + e^{-i\underline{l}\cdot(\underline{y}-\underline{a})} - e^{-i\underline{l}\cdot\underline{y}}\right) =$$

$$= \frac{1}{S_A}\int \frac{d^2 l}{l^4}\Big[4 - 2e^{i\underline{l}\cdot\underline{a}} - 2e^{-i\underline{l}\cdot\underline{a}} + 2e^{i\underline{l}\cdot(\underline{x}-\underline{y}+\underline{a})} - 2e^{i\underline{l}\cdot(\underline{x}-\underline{y})} +$$

$$+ 2e^{-i\underline{l}\cdot(\underline{x}-\underline{y}+\underline{a})} - 2e^{-i\underline{l}\cdot(\underline{x}-\underline{y})}\Big] = \frac{1}{S_A}\, 2\pi \int_0^\infty \frac{dl}{l^3}\Big[4 - 4J_0(la) +$$

$$+ 4J_0(l|\underline{x}-\underline{y}+\underline{a}|) - 4J_0(|\underline{l}|\cdot|\underline{x}-\underline{y}|)\Big] \approx \frac{2\pi}{S_A}\int_{1/a}^{1/|\underline{x}-\underline{y}|} \frac{dl}{l^3}\Big[\cancel{l^2 a^2 - l^2|\underline{x}-\underline{y}+\underline{a}|^2} +$$

as $|\underline{x}-\underline{y}| \ll a$

$$+ |\underline{x}-\underline{y}|^2 l^2\Big] = \frac{2\pi}{S_A}\; |\underline{x}-\underline{y}|^2 \int_{1/a}^{1/|\underline{x}-\underline{y}|}\frac{dl}{l} = \frac{2\pi}{S_A}\,|\underline{x}-\underline{y}|^2\cdot \ln\frac{a}{|\underline{x}-\underline{y}|}.$$

$$\Rightarrow \left\langle \ln^2\left(\frac{|\underline{x}-\underline{x}_1|}{|\underline{x}-\underline{x}_1'|}\,\frac{|\underline{y}-\underline{x}_1'|}{|\underline{y}-\underline{x}_1|}\right)\right\rangle = \frac{2\pi}{S_A}\,|\underline{x}-\underline{y}|^2 \ln\frac{a}{|\underline{x}-\underline{y}|}$$

Iterating the procedure for all the nucleons we get

$$C_F N_c \left[1 - \frac{\overset{=(4\pi\alpha)^2}{g^4}}{16\pi S_A} |\underline{x}-\underline{y}|^2 \ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}\right]^{j-1} \simeq$$

$$\simeq C_F N_c\, e^{-\frac{\pi\alpha^2}{S_A}|\underline{x}-\underline{y}|^2 \ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}\cdot(j-1)}$$

Such that

$$\varphi(x,k_T^2) = \frac{g^2}{2\pi S_A N_c}\cdot \ln\frac{a^2}{|\underline{x}-\underline{y}|^2}\cdot\int\frac{d^2x\, d^2y}{(2\pi)^2}\, e^{-i\underline{k}\cdot(\underline{y}-\underline{x})}\cdot C_F N_c$$

$$\cdot \underbrace{\sum_{j=1}^{A} e^{-\frac{\pi\alpha^2}{S_A}(j-1)|\underline{x}-\underline{y}|^2 \ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}}}$$

Rewrite as $\displaystyle \frac{A}{2\sqrt{R^2-b^2}}\int_0^{2\sqrt{R^2-b^2}} dz \cdot e^{-\frac{\pi\alpha^2}{V_A} zA|\underline{x}-\underline{y}|^2 \ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}} =$

$$= +\frac{V_A}{\pi\alpha^2 A}\,\frac{1}{|\underline{x}-\underline{y}|^2\ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}}\left[1 - e^{-\frac{\pi\alpha^2}{S_A}\cdot A\,|\underline{x}-\underline{y}|^2\ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}}\right].$$

$$\cdot\frac{\cancel{A}}{2\sqrt{R^2-b^2}} = \frac{S_A}{\pi\alpha^2}\,\frac{1}{|\underline{x}-\underline{y}|^2\ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}}\left[1 - e^{-\frac{\pi\alpha^2}{S_A} A\,|\underline{x}-\underline{y}|^2\ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}}\right].$$

giving

$$\varphi(x,k_T^2) = \int \frac{\cancel{2\alpha}}{\cancel{S_A N_c}}\,\cancel{\ln\frac{a^2}{|\underline{x}-\underline{y}|^2}}\cdot\frac{\cancel{S_A}}{\cancel{2}\pi\alpha^2}\,\frac{\cancel{C_F}\,\cancel{N_c}}{|\underline{x}-\underline{y}|^2\cancel{\ln\frac{|\underline{x}-\underline{y}|^2}{a^{-2}}}}\cdot\frac{d^2x\, d^2y}{(2\pi)^2}\, e^{-i\underline{k}\cdot(\underline{x}-\underline{y})}.$$

$$\cdot \left[1 - e^{-\frac{\pi \alpha^2}{S_A} A \, |\underline{x} - \underline{y}|^2 \ln \frac{|\underline{x}-\underline{y}|^2}{a^2}} \right] \Rightarrow \left| \begin{array}{l} \underline{b} = \frac{1}{2}(\underline{x} + \underline{y}) \\ \underline{r} = \underline{x} - \underline{y} \end{array} \right. =$$

$$= \frac{C_F}{2\pi^3 \alpha_s} \int d^2 b \, d^2 r \, e^{-i \underline{k} \cdot \underline{r}} \frac{1}{\underline{r}^2} \left[1 - e^{-\frac{\pi \alpha^2}{S_A} A \, \underline{r}^2 \ln \frac{\underline{r}^2}{a^2}} \right]$$

Defining the <u>saturation scale</u>

$$\boxed{Q_s^2 \equiv \frac{8 \pi \alpha^2}{S_A} A} \qquad \left(\text{note: } Q_s^2 \sim A^{1/3} \text{ as expected} \right)$$

we obtain:

$$\boxed{\varphi(x, k_T^2) = \frac{C_F}{2\pi^3 \alpha_s} \cdot \int d^2 b \, d^2 r \, e^{-i \underline{k} \cdot \underline{r}} \frac{1}{\underline{r}^2} \left[1 - e^{-\frac{1}{4} r_T^2 Q_s^2 \ln \frac{1}{r_T \Lambda}} \right]}$$

where we replaced $\frac{1}{a} \to \Lambda$.

Let's study the obtained $\varphi(x, k_T^2)$:

(a) $k_T \gg Q_s \Rightarrow \quad \varphi \simeq \frac{C_F}{2\pi^3 \alpha_s} \cdot \frac{1}{4} Q_s^2 \, S_A \int d^2 r \, e^{-i \underline{k} \cdot \underline{r}} \cdot \ln \frac{1}{r_T \Lambda} =$

$$= \frac{C_F}{8\pi^3 \alpha_s} Q_s^2 S_A \frac{2\pi}{k_T^2} = \underbrace{\frac{C_F \, Q_s^2 \, S_A}{(2\pi)^2 \alpha_s}}_{\frac{2\alpha C_F}{\pi}} \frac{1}{k_T^2} \sim \frac{1}{k_T^2} \text{ just like}$$

lowest order  distribution.

(b) $k_T \ll Q_s$

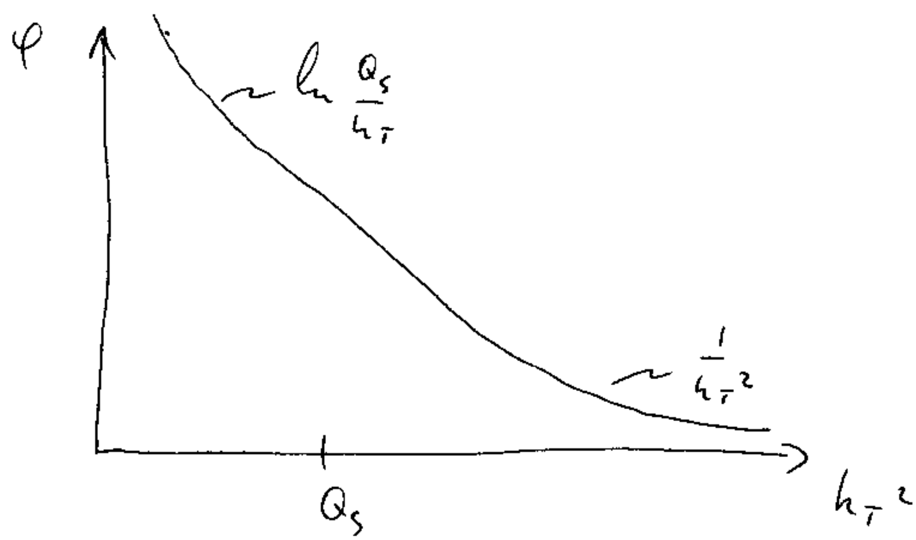
$$\varphi \simeq \frac{C_F}{2\pi^3 \alpha_s} S_A \cdot \int_{1/Q_s} \frac{d^2 r}{r^2} e^{-i \underline{k} \cdot \underline{r}} = \frac{C_F}{2\pi^3 \alpha_s} S_A \cdot 2\pi \cdot \ln \frac{Q_s}{k_T}$$

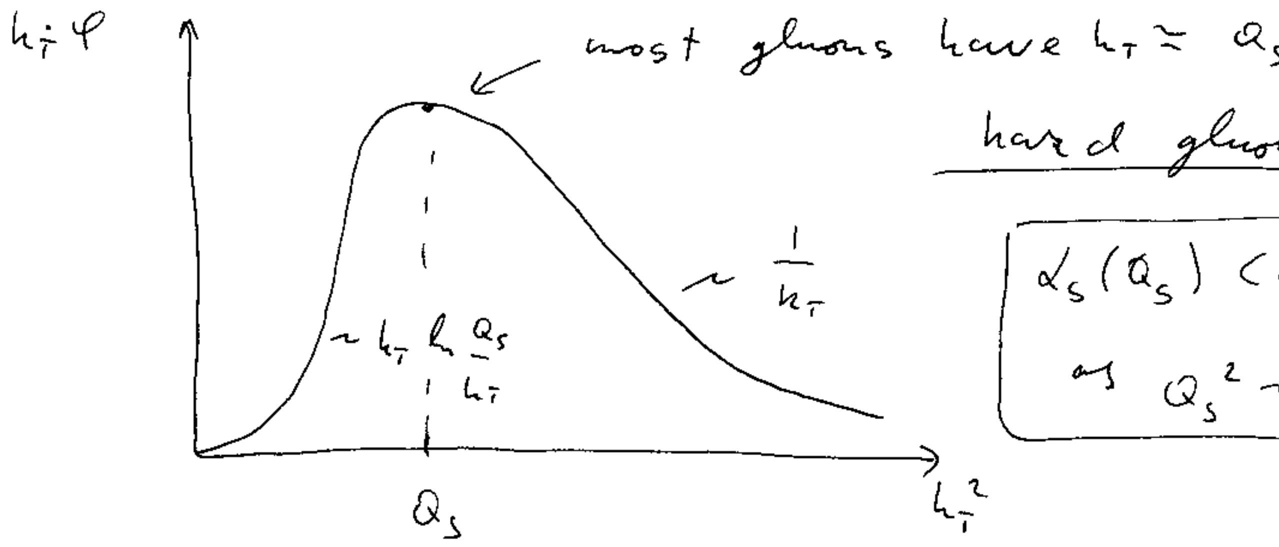
$$\Rightarrow \boxed{\varphi \simeq \frac{C_F}{\pi^2 \alpha_s} S_A \ln \frac{Q_s}{k_T}}$$

~ much softer IR divergence

$\left(\sim \frac{1}{k_T^2} \to \ln \frac{Q_s}{k_T} \right)$

$\Rightarrow$ classical field regulates the IR problem!





<u>hard gluons!</u>

$$\boxed{\begin{array}{l} \alpha_s(Q_s) \ll 1 \\ \text{as } Q_s^2 \sim A^{1/3} \end{array}}$$