Extendable Cooperative Games

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Abstract

A (TU) cooperative game is extendable if every core allocation of each subgame can be extended to a core allocation of the game. It is strongly extendable if any minimal vector in the upper core of any of its subgames can be extended to a core allocation. We prove that strong extendability is equivalent to largeness of the core. Further, we characterize extendability in terms of an extension of the balanced cover of the game. It is also shown how this extension can unify the analysis of many families of games under one roof.

1. Introduction

The purpose of this note is twofold. It is first intended to study a property of TU cooperative games (games, henceforth) called extendability. A game is *extendable* if every core allocation of each subgame can be extended to a core allocation of the game. This concept originates in an unpublished paper of Kikuta and Shapley (1986) and was further investigated by van Gellekom, Potters, and Reijnierse (1999). Our main result characterizes extendable games. Furthermore, we introduce a stronger notion of extendability and show that it is equivalent to largeness of the core (Sharkey 1982).

Extendability is an important property when the set of players is dynamic and different players join the game at different times. Assume that a group of veterans is involved in a game-like situation and that a core allocation has been established. When a group of rookies is joining the game a core allocation of the expanded game needs to be chosen. If the new game is not extendable, then it might be necessary to change the allocation of the veterans, possibly

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reducing the payoffs to some of them. Extendability guarantees that one can find a new core allocation that does not harm any of the veterans.

Our second aim is to show how various families of games can be characterized in terms of a certain function, called the *concavification* of the game. Viewed geometrically, the domain of a game v is the vertices of the unit cube. Rather than restricting attention to the vertices, we consider the entire cube and refer to the minimal concave and homogeneous function which is greater than or equal to v.[1] This function coincides with the totally balanced cover of v on the vertices of the cube. Balancedness and totally balancedness, convexity, exactness, largeness of the core, and extendability can all, quite effortlessly, be characterized in terms of this extension. As a consequence, the hierarchy among these properties becomes clearly apparent.

Since our extension of the balanced cover is homogeneous it is uniquely determined by its values on the unit simplex. We find it more convenient to restrict attention to the simplex rather than considering the entire cube. We therefore, identify each coalition with the uniform distribution over its members and not with the corresponding characteristic vector. The characteristic function describes the *per capita* value of each coalition rather than the total worth of a coalition.

This "normalized version" of a game might be seen dispensable. However, two arguments justify its use. First, when working in the simplex the point corresponding to the grand coalition becomes an interior point of the domain. This fact is important in the proof of our characterization of extendability (Theorem 1). Second, the dimension of the domain is reduced by 1. This makes the geometry simpler and the proofs more visible. For instance, Lemma 2, which is essentially the Shapley–Bondareva theorem (Shapley 1967, Bondareva 1962), becomes very intuitive when presented in this way.[2]

In the following section we provide an example to motivate the study of the extendability property. In Section 3, we formally define the normalized game and the extension of the balanced cover to the unit simplex. Section 4 discusses the properties of convexity, exactness, and largeness of the core. Some of the results in this section are variants of already known facts. The proofs are omitted whenever this is the case. In Section 5, extendability is investigated. Some of the proofs are deferred to the Appendix.

2. Pricing in Public Enterprizes: An Example

The following application of cooperative game theory is adopted from Faulhaber (1975). A publicly owned (or a privately owned and publicly regulated) enterprize produces and sells a set T of different products. Possible

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[1]This is not a new idea. Similar constructions can be found in Lovasz (1983), Ichiishi (1990), and Weber (1994).

[2]Of course, intuition is a subjective matter. Still, we believe that many will consider our proof as more intuitive than those standard in the literature.

examples include a railway company selling tickets to various destinations, a communication company providing long- and short-distance calls as well as internet connection, and a dam maintenance company providing flood control, water for irrigation, and electric power. For every subset $R \subseteq T$, the cost of providing the required demands only for the products in R is given by $v(R)$.

For equity reasons, public policy makers would like to design a pricing scheme that keeps a *balanced budget* and is *subsidy-free*. The first means that the total revenue, $\sum_{i \in T} x_i$, is equal to the total cost, $v(T)$. The latter means that $\sum_{i \in R} x_i \leq v(R)$ for every $R \subseteq T$. This implies that no subset of products subsidies the rest. In terms of cooperative game theory the vector $\{-x_i\}_{i \in T}$ is in the core of the game $-v$.

Now, assume that the enterprize considers an expansion of its services and to provide a new set of products, S, in addition to T. The domain of the function v is extended to all subsets of $N := T \cup S$.

The notion of extendability, which is the main theme of this note, refers to the relation between the original v, defined over the restricted domain, T, and its extension to the grand domain, N. When the new game $(N, -v)$ is *extendable* it is possible to find a pricing scheme $\{y_i\}_{i \in N}$ in the core of $(N, -v)$ (i.e., $\sum_{i \in N} y_i = v(N)$, $\sum_{i \in R} y_i \leq v(R)$ for every $R \subseteq N$) such that $y_i = x_i$ for every $i \in T$. In other words, extendability of $-v$ enables one to introduce a pricing scheme for the large set of products that maintains the balanced budget and subsidy-free properties, while keeping the prices of the products in T unchanged. In particular, expanding the menu of services provided does not force a price increase of any of the original services in T.

On the other hand, if the game $(N, -v)$ is *not* extendable, then it might be that every core allocation y of this game has some product $i \in T$ that needs to become more expensive (i.e., $y_i > x_i$). Policy makers will have to raise the price of some products in T, which is likely to lead consumers of the products that are about to become more expensive to object to the proposed expansion. This might make the reform harder to implement. This is why extendability is an essential characteristic of a game.

3. Preliminaries

A game with player set N ($|N| = n$) is a function $v : 2^N \to I\!R$ with $v(\emptyset) = 0$. If $S \subseteq N$ is a nonempty coalition then the subgame of v, with respect to the coalition S, is a cooperative game v_S, where the grand coalition is S and $v_S(T) = v(T)$ for every $T \subseteq S$. We denote by $C(v)$ the *core* of the game v, that is $C(v) = \{x \in I\!R^N;\ x(N) = v(N),\ x(S) \geq v(S)$ for every $S \subseteq N\}$. The *upper core* of a game v, denoted $U(v)$, is the set $U(v) = \{x \in I\!R^N;\ x(S) \geq v(S)$ for every $S \subseteq N\}$.

A balanced collection for a coalition $S \subseteq N$ is a set of coalitions $\{T_1, T_2, \ldots, T_k\}$ and nonnegative numbers $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ such that

$\sum_{i=1}^{k} \alpha_i e_{T_i} = e_S$, where, for every coalition $R \subseteq N$, $e_R \in I\!R^N$ is the vector whose i'th coordinate equals 1 if $i \in R$ and 0 otherwise. For a given game v, the *totally balanced cover* of v is the game $\bar{v}$ (with the same player set as v) defined by $\bar{v}(S) = \max \sum_{i=1}^{k} \alpha_i v(T_i)$, where the maximum is taken over all balanced collections for the coalition S. A game v is called *balanced* if $\bar{v}(N) = v(N)$ and *totally balanced* if $\bar{v}(S) = v(S)$ for every $S \subseteq N$. It is well known (see Shapley 1967, Bondareva 1962) that v has a nonempty core iff it is balanced and that every subgame of v has a nonempty core iff v is totally balanced.

We denote by Δ the unit simplex of $I\!R^N$, that is $\Delta = \{(q^1, \dots, q^n); \sum_{i=1}^{n} q^i = 1, q^i \geq 0, i = 1, \dots, n\}$. For $S \subseteq N$, $\Delta_S \subseteq \Delta$ is the set of distributions over N whose support is contained in S. For any nonempty coalition $S \subseteq N$ denote $c_S = \frac{1}{|S|} e_S \in \Delta$. Given a game v, let $\hat{v}(c_S) = \frac{v(S)}{|S|}$ for every nonempty $S \subseteq N$. $\hat{v}$ is a function over a set of $2^n - 1$ points in the n-dimensional unit simplex. $\hat{v}(c_S)$ is the average worth of the coalition S, or the percapita value of the members of S.

DEFINITION 1: *The concavification of $\hat{v}$, $\mathbf{cav}\hat{v}$, is defined as the minimum of all concave functions $f : \Delta \to I\!R$ such that $f(c_S) \geq \hat{v}(c_S)$ for every nonempty coalition S.*

Remark 1: Since the minimum of a family of concave functions over Δ is concave, $\mathbf{cav}\hat{v}$ is concave. Thus, $\mathbf{cav}\hat{v}$ is the minimal concave function that is greater than or equal to $\hat{v}$ on every point of the type c_S.

LEMMA 1: *For every $q \in \Delta$,*

$$\mathbf{cav}\hat{v}(q) = \max \left\{ \sum_{S \subseteq N} \alpha_S \hat{v}(c_S); \sum_{S \subseteq N} \alpha_S c_S = q, \alpha_S \geq 0; \text{ and } \sum_{S \subseteq N} \alpha_S = 1 \right\}.$$

Proof: Denote $w(q) = \max\{\sum_{S \subseteq N} \alpha_S \hat{v}(c_S); \sum_{S \subseteq N} \alpha_S c_S = q, \alpha_S \geq 0;$ and $\sum_{S \subseteq N} \alpha_S = 1\}$. Since w is concave and $\hat{v} \leq w$, $\mathbf{cav}\hat{v} \leq w$. On the other hand, if $\sum_{S \subseteq N} \alpha_S c_S = q$ where $\alpha_S \geq 0$ and $\sum_{S \subseteq N} \alpha_S = 1$, then by concavity of $\mathbf{cav}\hat{v}$, $\mathbf{cav}\hat{v}(q) \geq \sum_{S \subseteq N} \alpha_S \mathbf{cav}\hat{v}(c_S) \geq \sum_{S \subseteq N} \alpha_S \hat{v}(c_S)$. Thus, $\mathbf{cav}\hat{v} \geq w$. ■

Remark 2: In view of Lemma 1, $\mathbf{cav}\hat{v}$ can be thought of as a generalization of the totally balanced cover of v (up to multiplication by the size of the relevant coalition). In particular, for every coalition $S \subseteq N$, $\mathbf{cav}\hat{v}(c_S) = \frac{\bar{v}(S)}{|S|}$.

DEFINITION 2: *For a function $f : \Delta \to \mathbb{R}$ and a point $p \in \Delta$, a vector $x \in I\!R^n$ is a linear support of f at p, if*[3] *$x \cdot p = f(p)$ and $x \cdot q \geq f(q)$ for any $q \in \Delta$.*

[3]For every two vectors $a = (a^1, \dots, a^n)$ and $b = (b^1, \dots, b^n)$ in $I\!R^n$, $a \cdot b$ denotes the inner product $a \cdot b = \sum_{i=1}^{n} a^i b^i$.

LEMMA 2:

(a) $C(v) \neq \emptyset$ *iff* $\mathbf{cav}\hat{v}(c_N) = \hat{v}(c_N)$.

(b) If v *is balanced then* $C(v)$ *is equal to the set of linear supports for* $\mathbf{cav}\hat{v}$ *at the point* c_N.

Proof: To prove (a), assume first that v has a nonempty core and let $x \in C(v)$. Consider the linear (and in particular concave) function on Δ defined by $f(q) = x \cdot q$. Since x is in the core, for every nonempty coalition $S \subseteq N$, $f(c_S) = x \cdot c_S = \frac{x(S)}{|S|} \geq \frac{v(S)}{|S|} = \hat{v}(c_S)$. It follows that $f(q) \geq \mathbf{cav}\hat{v}(q)$ for every $q \in \Delta$. By a similar argument, $f(c_N) = x \cdot c_N = \hat{v}(c_N)$. Therefore, $\hat{v}(c_N) \leq \mathbf{cav}\hat{v}(c_N) \leq x \cdot c_N = \hat{v}(c_N)$.

In order to prove the inverse direction, assume that $\mathbf{cav}\hat{v}(c_N) = \hat{v}(c_N)$. Since $\mathbf{cav}\hat{v}$ is concave, it has a linear support, say x, at the point c_N. By assumption, $x \cdot c_N = \mathbf{cav}\hat{v}(c_N) = \hat{v}(c_N)$. Also, for every $S \subseteq N$, $\frac{x(S)}{|S|} = x \cdot c_S \geq \hat{v}(c_S) = \frac{v(S)}{|S|}$, which implies $x(S) \geq v(S)$. Therefore, $x \in C(v)$ which proves (a). (b) follows from the proof of (a). ■

Remark 3: A similar characterization of balanced games appears in Branzei and Tijs (2001). Here, however, it is related to the concavification operator.

4. Convexity, Exactness, and Large Cores

4.1. Convex Games

DEFINITION 3 *(Shapley 1971): The game* v *is convex if for any two coalitions* S *and* T, $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$.

A first characterization of convex games via $\mathbf{cav}\hat{v}$ appears in the following proposition. Weber (1994, Section 9, pp. 1295–1297) has a similar result. We omit the proof.

PROPOSITION 1: *A game* v *is convex iff, for every pair of nonempty coalitions* S, T *such that* $T \subseteq S$, *the line segment connecting the points* $(c_S, \hat{v}(c_S))$ *and* $(c_T, \hat{v}(c_T))$ *is on the graph of* $\mathbf{cav}\hat{v}$.

We next provide another characterization of convex games. This is done by an explicit description of $\mathbf{cav}\hat{v}$ when the game is convex. The proof of the proposition can be rather easily obtained from the results of Delbaen (1974, lemma 2, pp. 214–215) or Lovasz (1983, pp. 246–249) and is therefore omitted. We first need some notation.

NOTATION 1:

(a) For a permutation π *over* N, *denote by* S_π^i *the coalition* $\{\pi(1), \pi(2), \ldots, \pi(i)\}$, $i = 1, \ldots, n$.

(b) *Let v be a game and π an order over the set of players N. Then the vector of marginal contributions, with respect to π is $x_\pi = (x_\pi^1, \ldots, x_\pi^n)$, where $x_\pi^i = v(S_\pi^i) - v(S_\pi^{i-1})$, $i \in N$.*

(c) *Let $q = (q^1, \ldots, q^n) \in \Delta$. π_q denotes a permutation of the players such that $q^{\pi_q(1)} \geq q^{\pi_q(2)} \geq \cdots \geq q^{\pi_q(n)}$. When there is more than one such permutation, i.e., $q^i = q^j$* for *some $i \neq j, \pi_q$ is any one of them.*

(d) *For $q = (q^1, \ldots, q^n) \in \Delta$, let $S_q^i = S_{\pi_q}^i$ and $x_q = x_{\pi_q}$.*

PROPOSITION 2: *A game v is convex iff $\mathbf{cav}\hat{v}(q) = q \cdot x_q$ for every $q \in \Delta$.*

4.2. Exact Games

DEFINITION 4 *(Schmeidler 1972): The game v is exact if, for every coalition S, there is $x \in C(v)$ such that $v(S) = x(S)$.*

PROPOSITION 3: *A game v is exact iff, for every nonempty coalition S, the line segment connecting the points $(c_S, \hat{v}(c_S))$ and $(c_N, \hat{v}(c_N))$ is on the graph of $\mathbf{cav}\hat{v}$.*

Proposition 3 is a variant of a result in Weber (1994, section 9, pp. 1295–1297). The proof is omitted. A simple consequence is the following corollary.

COROLLARY 1: *Let v be a totally balanced game. Then v is exact iff $\mathbf{cav}\hat{v}(c_S) = \min\{x \cdot c_S; x \in C(v)\}$ for every $S \subseteq N$.*

4.3. Games with a Large Core

DEFINITION 5 *(Sharkey 1982): v has a large core if for every $y \in U(v)$ there exists $x \in C(v)$ such that $x \leq y$.*

PROPOSITION 4: *v has a large core iff $\mathbf{cav}\hat{v}(q) = \min\{x \cdot q; x \in C(v)\}$ for every $q \in \Delta$.*

Proposition 4 will follow from Lemma 4 which is a consequence of Lemma 3. The proofs of these lemmas are deferred to the Appendix. In Ichiishi (1990, proposition 2.4) a similar characterization of games with *no-gap* (the dual concept of large core) appears.

LEMMA 3: *Let $A \subseteq \mathbb{R}^n$ be a convex set and let $f : A \to \mathbb{R}$ be concave. Assume that $H \subseteq \mathbb{R}^n$ has the following two properties:*

(i) *H is closed and convex.*

(ii) For every $q \in A$ there is $y \in H$ such that $y \cdot q = f(q)$ and $y \cdot q' \geq f(q')$ for every $q' \in A$ (the hyperplanes defined by the vectors in H support the entire graph of f).

Let q_0 be in the interior of A and assume that $x \in I\!R^n$ is a linear support for f at q_0. Then $x \in H$.

LEMMA 4: *Let $A = \Delta$, and let f and H be as in the previous lemma. Assume that q_0 is in the relative interior of Δ and that x is a linear support for f at q_0. Then, $x \in H$.*

Proof of Proposition 4: Assume first that v has a large core and fix $q \in \Delta$. By Lemma 2 (b), if $x \in C(v)$ then $x \cdot q \geq \mathbf{cav}\hat{v}(q)$. Therefore, $\mathbf{cav}\hat{v}(q) \leq \min\{x \cdot q; x \in C(v)\}$. Now, let y be a linear support of $\mathbf{cav}\hat{v}$ at the point q. Then, $y \in U(v)$ and, since v has a large core, there is $x \in C(v)$ such that $x \leq y$. Since x is in $C(v)$, $\mathbf{cav}\hat{v}(q) \leq x \cdot q$ and since $x \leq y$, $\mathbf{cav}\hat{v}(q) = y \cdot q \geq x \cdot q$. It follows that $x \cdot q = \mathbf{cav}\hat{v}(q)$, and therefore, $\mathbf{cav}\hat{v}(q) = \min\{x \cdot q; x \in C(v)\}$.

Conversely, assume that $\mathbf{cav}\hat{v}(q) = \min\{x \cdot q; x \in C(v)\}$ and let $y \in U(v)$. Obviously, $y \cdot q \geq \mathbf{cav}\hat{v}(q)$ for every $q \in \Delta$. Therefore, there is $x \leq y$ such that x is a linear support of $\mathbf{cav}\hat{v}$ at some point $q \in \Delta$. Moreover, x can be chosen in a way that it supports $\mathbf{cav}\hat{v}$ at a point q in the relative interior of Δ. We can thus apply Lemma 4, where (in the lemma's notation) $A = \Delta$, $f = \mathbf{cav}\hat{v}$, $H = C(v)$, and $q_0 = q$. It follows that $x \in C(v)$ and $x \leq y$. ∎

5. Extendable Games

5.1. Extendability

We denote by $C_S(v)$ the projection of $C(v)$ to the subspace corresponding to the coalition S. That is, $C_S(v) = \{x \in I\!R^S$; there is $y \in C(v)$ such that $y^i = x^i$ for all $i \in S\}$.

DEFINITION 6: *v is extendable if $C(v_S) \subseteq C_S(v)$ for every $S \subseteq N$.*

Extendability was introduced by Kikuta and Shapley (1986) and named by van Gellekom, Potters, and Reijnierse (1999).

THEOREM 1: *Let v be a totally balanced game. Then, v is extendable iff there is $\delta > 0$ such that if, for some $S \subseteq N$, $q \in \Delta_S$ satisfies $|q - c_S| < \delta$, then $\mathbf{cav}\hat{v}(q) = \min\{x \cdot q; x \in C(v)\}$.*

Proof: Assume first that v is extendable. Since $\mathbf{cav}\hat{v}$ is piecewise linear, for every $S \subseteq N$ one can find a small enough number $\delta_S > 0$ such that for every $q \in \Delta_S$ with $|q - c_S| < \delta_S$ and for every $0 \leq \alpha \leq 1$, $\alpha \mathbf{cav}\hat{v}(q) + (1-\alpha)\mathbf{cav}\hat{v}(c_S) = \mathbf{cav}\hat{v}(\alpha q + (1-\alpha)c_S)$. Let $\delta = \min\{\delta_S;\ S \subseteq N\}$. Fix $S \subseteq N$ $q \in \Delta_S$ with $|q - c_S| < \delta$. Then, by the separation theorem, there is $x \in I\!R^S$ which is a linear support of the restriction of $\mathbf{cav}\hat{v}$ to Δ_S at both points c_S and q. Since v is totally balanced, and by Lemma 2 (b), $x \in C(v_S)$. Thus, by assumption, we can extend x to a core vector of the game v, say $\tilde{x}$. Obviously, $\tilde{x} \cdot q = x \cdot q = \mathbf{cav}\hat{v}(q)$.

Conversely, assume that $x \in C(v_S)$ for some $S \subseteq N$. Then, x is a linear support of the restriction of $\mathbf{cav}\hat{v}$ to Δ_S at the point c_S. The set $C_S(v)$ is closed and convex. Moreover, by assumption, for every $q \in \Delta_S$ sufficiently close to c_S there is $y \in C_S(v)$, which is a linear support of the restriction of $\mathbf{cav}\hat{v}$ to Δ_S at q. Thus, we may apply Lemma 4 and deduce that $x \in C_S(v)$. ■

Remark 4: The second part of the proof does not use the assumption that v is totally balanced. It follows that any game v which satisfies the condition of the theorem is extendable. Thus, by Proposition 4, large core implies extendability.

5.2. Strong Extendability

DEFINITION 7: $y \in U(v_S)$ *is* minimal if $x \in U(v_S)$ *and* $x \leq y$ *implies* $y = x$.

DEFINITION 8: v *is* strongly extendable *if, for every coalition S and a minimal y* $\in U(v_S), y \in C_S(v)$.

LEMMA 5: *If v is strongly extendable, then it is extendable.*

Proof: This is obvious since if $y \in C(v_S)$, then it is minimal in $U(v_S)$. ■

THEOREM 2: *v is strongly extendable iff it has a large core.*

Proof: Suppose that v has a large core and let $y \in U(v_S)$ be minimal. One can extend y to a vector, say z, in $U(v)$. Since v has a large core, there is $x \in C(v)$ such that $x \leq z$. Thus, x_S, the restriction of x to Δ_S, satisfies $x_S \leq y$. By the minimality of y, $x_S = y$.

As for the converse, assume that v is strongly extendable. We use Proposition 4 and show that $\mathbf{cav}\hat{v}(q) = \min\{x \cdot q; x \in C(v)\}$ for every $q \in \Delta$. Since $\mathbf{cav}\hat{v}$ is concave, it is sufficient to show that for every $S \subsetneqq N$ and for every $q \in \Delta_S$, $\mathbf{cav}\hat{v}(q) = \min\{x \cdot q; x \in C(v)\}$. Moreover, since $\mathbf{cav}\hat{v}$ is continuous, it is sufficient to show it for every $S \subsetneqq N$ and for every point q in the relative interior of Δ_S.

Let $S \subsetneq N$ and q be a relative interior point of Δ_S (i.e., $q_i > 0$ for every $i \in S$). The function $\mathbf{cav}\hat{v}_S$ is concave, and as such it has a support at q. Denote this support by y. In particular, $y \in U(v_S)$ and $y \cdot q = \mathbf{cav}\hat{v}_S(q)$. Since all the coordinates of q are positive, it implies that y is minimal. By assumption, $y \in C_S(v)$. It means that there is $x \in C(v)$ whose restriction to Δ_S coincides with y. Thus, $x \cdot q = y \cdot q = \mathbf{cav}\hat{v}_S(q)$. This shows that $\min\{x \cdot q; x \in C(v)\} \leq \mathbf{cav}\hat{v}_S(q)$. Since always $\min\{x \cdot q; x \in C(v)\} \geq \mathbf{cav}\hat{v}_S(q)$, the desired equality is proven. ■

Appendix

Proof of Lemma 3: Assume, contrary to the lemma, that $x \notin H$. Since H is closed and convex, and by the separation theorem, there is $0 \neq p \in \mathbb{R}^n$ such that $p \cdot x > \max\{p \cdot y; y \in H\}$. Since q_0 is in the interior of A, one can find a small enough $\delta > 0$ such that $q_0 - \delta p \in A$. Let $y \in H$ be a linear support of f at q_0 and let $z \in H$ be a linear support for f at $q_0 - \delta p$. Then,

$$z \cdot (q_0 - \delta p) = f(q_0 - \delta p) \leq x \cdot (q_0 - \delta p) < y \cdot (q_0 - \delta p). \quad \text{(A1)}$$

It follows that there is $0 < \alpha \leq 1$ such that

$$\alpha z \cdot (q_0 - \delta p) + (1 - \alpha) y \cdot (q_0 - \delta p) = x \cdot (q_0 - \delta p). \quad \text{(A2)}$$

Denote $w = \alpha z + (1 - \alpha) y \in H$. It follows that

$$\begin{aligned} w \cdot (q_0 - \delta p) &= w \cdot q_0 - \delta w \cdot p > w \cdot q_0 - \delta x \cdot p \\ &\geq f(q_0) - \delta x \cdot p = x \cdot (q_0 - \delta p). \end{aligned} \quad \text{(A3)}$$

However, (A2) and (A3) contradict each other. Therefore, $x \in H$. ■

Proof of Lemma 4: Consider the homogeneous extension of f to $\mathbb{R}^n_+$, denoted $\hat{f}$. That is, $\hat{f}(a) = |a| f(\frac{a}{|a|})$ for every $a \in \mathbb{R}^n_+$ ($|a| = \sum_{i=1}^n |a^i|$ is the l_1 norm of a). Since f is concave on Δ, it is straight forward that $\hat{f}$ is concave on $\mathbb{R}^n_+$. Also, for every $a \in \mathbb{R}^n_+$, if y is a linear support of f at $\frac{a}{|a|}$, then it is a linear support of $\hat{f}$ at a. Finally, since q_0 is in the relative interior of Δ it is an interior point of $\mathbb{R}^n_+$. Thus, we can apply Lemma 3 and deduce that $x \in H$. ■

References

BONDAREVA, O. (1962) The theory of the core in an n-person game (in Russian), *Vestnik Leningrad. Univ.* **13**, 141–142.

BRANZEI, R., and S. H. TIJS (2001) Cooperative games with a simplicial core, *Balkan Journal of Geometry and Applications* **6**, 7–15.

DELBAEN, F. (1974) Convex games and extreme points, *Journal of Mathematical Analysis and Applications* **45**, 210–233.

FAULHABER, R. F. (1975) Cross-subsidization: Pricing in public enterprises, *The American Economic Review* **65**, 966–977.

ICHIISHI, T. (1990) Comparative cooperative game theory, *International Journal of Game Theory* **19**, 139–152.

KIKUTA, K., and L. S. SHAPLEY (1986) Core stability in n-person games, Unpublished paper.

LOVASZ, L. (1983) Submodular functions and convexity, in *Mathematical Programming: The State of the Art*, A. Bachem et al, eds. Berlin: Springer-Verlag, 235–257.

SCHMEIDLER, D. (1972) Cores of exact games I, *Journal of Mathematical Analysis and Applications* **40**, 214–255.

SHAPLEY, L. S. (1967) On balanced sets and cores, *Naval Research Logistics Quarterly* **14**, 453–460.

SHAPLEY, L. S. (1971) Cores of convex games, *International Journal of Game Theory* **1**, 11–26.

SHARKEY, W. W. (1982) Cooperative games with large cores, *International Journal of Game Theory* **11**, 175–182.

VAN GELLEKOM, J. R. G., J. A. M. POTTERS, and J. H. REIJNIERSE (1999) Prosperity properties of TU-games, *International Journal of Game Theory* **28**, 211–227.

WEBER, R. J. (1994) Games in coalitional form, in *Handbook of Game Theory, Vol. II*. R. J. Aumann and S. Hart, eds. Amsterdam: Elsevier.