

Understanding probabilistic interpretations of physical systems: A prerequisite to learning quantum physics

Lei Bao

Department of Physics, The Ohio State University, 174 W 18th Avenue, Columbus, Ohio 43210

Edward F. Redish

Department of Physics, University of Maryland, College Park, Maryland 20742

(Received 31 July 2001; accepted 11 December 2001)

Probability plays a critical role in making sense of quantum physics, but most science and engineering undergraduates have very little experience with the topic. A probabilistic interpretation of a physical system, even at a classical level, is often completely new to them, and the relevant fundamental concepts such as the probability distribution and probability density are rarely understood. To address these difficulties and to help students build a model of how to think about probability in physical systems, we have developed a set of hands-on tutorial activities appropriate for use in a modern physics course for engineers. We discuss some student difficulties with probability concepts and an instructional approach that uses a random picture metaphor and digital video technology. © 2002 American Association of Physics Teachers.

[DOI: 10.1119/1.1447541]

I. INTRODUCTION

A student's first course in quantum physics can be quite difficult. They have to think about phenomena for which they have no direct personal experience, they have to follow long chains of inference from experiment to what appear to be bizarre conclusions, and they have to deal with phenomena that fundamentally involve probability. The latter concept introduces a number of difficulties. Students of physics are rarely introduced to the use of probability in classical situations early in their studies, even in places where it would be appropriate, such as error analysis or statistical mechanics. In addition, studies of students' understanding of probabilistic ideas in cognitive psychology¹ and mathematics education² research indicate that serious misunderstandings are common.

The Physics Education Research Group at the University of Maryland has studied the difficulties students have in learning quantum physics. The purpose of this paper is to discuss the highlights of this research with an emphasis on its practical values to instruction. Our research was carried out in two venues: the third semester of our introductory calculus-based engineering physics class (Physics 263), and an upper division one-semester course in quantum physics for engineers (Physics 420). The emphasis was mostly on the latter course, and most of our curriculum development was tested there. The 263 class is required of all engineering majors. The 420 class is an upper division elective for engineers so it is considerably smaller (15–30 students). It is dominated by electrical engineers (80–90%) and is taught every semester.

After years of experience in a seemingly deterministic world, reinforced by learning classical physics, students can develop a strong deterministic view of the physical world. In most classical situations discussed in introductory physics classes, the behavior of a physical system can be precisely determined, and the emphasis is often on the construction of a detailed description of the motion of an object.

In quantum mechanics, students have to use and interpret probabilistic representations that are very different from the deterministic ones they have become accustomed to thinking

of as physics. We first discuss the kinds of difficulties students encounter with probability, including the gambler's fallacy³ and the difficulty with the idea of a probability density. Then we consider instructional environments that can help students understand the fundamental concepts of probability and learn to use a probabilistic representation to interpret physical systems. To help students over the gaps, *bridges* are needed, and we make this bridging process in two steps.⁴ The first is to help students develop a basic understanding of probability using contexts with which they are familiar. The second is to use new metaphors, hands-on analysis, and video tools to bring the students from an understanding of classical probability to an understanding of quantum probability.

II. STUDENT DIFFICULTIES IN UNDERSTANDING PROBABILITY

Traditional instruction of quantum mechanics assumes that classical prerequisites such as the understanding of probability and energy diagrams are readily accessible to students. However, students often have much difficulty with these prerequisites.⁵ Specifically, we wanted to learn if the students were able to decipher the meaning of the phrase probability of locating a particle in a certain region. In general, most undergraduate students are familiar only with a kinematical description of motion (a particle trajectory observed over a period of time). They may find it difficult to comprehend how a probabilistic representation relates to actual observations and how the measurement can be used to construct details of the particle's behavior.

Our observations were conducted with students from two classes of Physics 263 (one in the fall semester of 1994 and one in the spring semester of 1996) and two classes of Physics 420 (one each in the spring and fall semesters of 1998). In the Physics 263 course, only the lecture section taught by one of the authors (EFR) was studied. For the Physics 263 courses, the class of fall 1994 used only traditional lectures and the class of spring 1996 used three quantum tutorials that addressed students' difficulties on classical prerequisites including classical probability.⁶ The Physics 420 class in the

spring of 1998 used quantum tutorials on both classical prerequisites and quantum issues and the class in the fall of 1998 was taught in a traditional fashion as a control by members of the department not participating in the research. Most of the instructional innovation used a tutorial format. Tutorials are a type of guided group-learning instruction developed by Lillian C. McDermott and co-workers at the University of Washington.⁷ The quantum tutorials were developed using a similar format.

The instruments used in this study to probe student thinking include two concept quizzes, one exam question, and student interviews. (The quiz and exam questions are given in the Appendix.) Question A were designed to probe students understanding of some fundamental ideas in probability including the independence of events and the gambler's fallacy. These problems were given at the beginning of the Physics 420 class to collect information on students initial understanding. Question B was designed to probe whether the students understand the different shapes of the wave function of bound states and if they could make the link between the amplitude of the wave function and the probability density of a particle being in certain region. The question also gave information on student understanding of the potential well. This question was given to the two Physics 263 classes after instruction.

Question C probes students understanding of probabilistic interpretations of both classical systems and quantum systems. This question was used in the final exam of the Physics 263 class in spring 1996.

A total of 16 individual interviews were conducted with students from the Physics 263 and 420 classes to investigate students understanding of the classical prerequisites. The part of the interview relating to probability was based on the same issues as probed by the quiz problems but with a more open-ended style. In the following, we briefly summarize our observations.

Predictability and the stochastic nature of probability: In the five interviews conducted with the Physics 263 students after instruction, we found that four of the students held a deterministic, empirical intuition of probability. (The five students all received a grade of A and are not representative of the overall population.) Their descriptions show an incorrect understanding of the difference between the stochastic nature of any single observation and the determined expected distribution of the results of ensemble observations. These students bring with them the belief that small samples will replicate the probabilistic trends expected from a very large number of trials and that the specific result of any single measurement can be affected by the previous sequence of outcomes.⁸ For example, one of the students responded to the first part of question A with *since you already have three heads in a row, you should have more chances to get a tail on the fourth time*.

A quantitative study of the students in the advanced class (Physics 420 with 18 students) in the spring of 1998 shows similar results. We used an open-ended survey with a problem about a coin-flipping experiment (see Appendix, question A.1). A majority of the students showed the gambler's fallacy: 61% thought that the result of a single coin-flipping event depends on the results of previous coin-flipping activity. In addition, 27% of the students thought that if the coin were flipped 100 times, there would be an exact 50/50 distribution for heads and tails. The last part of question A concerns probable values of students SAT scores and also deals

with the same issue. In this case more than 67% of the students thought that knowing one student's score would affect the probable average score of the other students.

Understanding probabilistic representations: None of the students who visited during our office hours or participated in interviews reported having had any experience (before instruction) using a probabilistic interpretation to think about a physical system. A very small number of them had the impression of doing some kind of probability analysis in a math class, but did not remember any details of the mathematics. None had used probability to describe a real physical event.

In the physics 263 class of fall 1994, we gave question B in a quiz after instruction in quantum mechanics. After the students were given the wave function and asked to determine where in the potential well the electron would most likely be found, most of them did not use the correct spatial dimension, x , in their reasoning. The largest fraction, 40%, left the question blank; 36% used the vertical dimension, V , as a spatial dimension for position.⁹ (As suggested by interview results, many of these students appeared to consider that electrons with different energy states would also be in different places on the vertical dimension in the potential well.) Only 9% of the class used the correct dimension and among them, only one student came up with the correct answer. Among all the students, only 11% gave some kind of reasoning for their answers.

III. UNDERSTANDING PROBABILITY WITH CLASSICAL SYSTEMS

The role of probability in microscopic systems is conceptually quite subtle. For most of the traditional experiments of quantum physics, it is not possible to set up an individual quantum object, for example, an atom, molecule, or nucleus, and probe it repeatedly.¹⁰ Instead, an ensemble of identically prepared objects is probed and the ensemble average is identified with the quantum average. Thus, in an $(e, 2e)$ experiment, thousands of electrons knock electrons out of thousands of different atoms or molecules, and for each individual case, the target electrons momenta before the collision are determined by momentum conservation. The result is interpreted as the probability distribution of finding a given momentum in a single atom or molecule.¹¹ Thus we note that *even if the fundamental mechanics of atoms and molecules were classical*, we would still need to describe most experiments with atoms using probabilities. This fact allows us to build a bridge to the use of probability in classical situations.

We introduce a metaphor, the *random picture*, as a fundamental tool for students to construct a probabilistic representation. Because atoms cannot be tracked or controlled individually, we ask students to consider a set of oscillating objects whose phases are random. We then ask the students to imagine taking a series of flash photographs of a single moving classical object at random times and using those photographs to predict where the object is most likely to be found. Based on this notion, hands-on activities and discussion questions were developed and used in tutorials where the students can apply this random picture metaphor to analyze real physical systems such as a cart moving back and forth on an air track.

Building the probability density function: Consider a simple classical system with periodic motion such as a pendulum bob swinging back and forth. The traditional approach

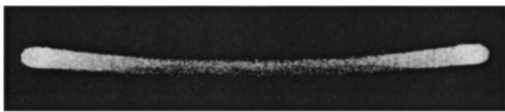


Fig. 1. Time exposure photo of a white pendulum bob swinging against a black background.

in classical mechanics is to think about the motion of the bob, the force on the bob during the motion, the velocity (or position) versus time relation, etc. Such approaches encourage students to focus on the motion of the objects, which encourages a deterministic view of physical systems.

Another way to analyze this system is to think about its probabilistic aspects. For example, if one does not know when the motion of the bob started, its position at an arbitrary time is uncertain. But one can still predict the probability of finding the bob in certain regions, even though the exact position–time relation of the bob is unknown. Figure 1 is a time-exposure photograph of a white pendulum bob swinging against a black background. The brightness of a particular area is a relative measure of the amount of time that the bob spends in the corresponding region. It therefore reflects the distribution of the probability density for the bob to be found at different areas.

The time-exposure photograph produces a continuous distribution function for the probability density. We can use the random picture idea to generate discrete measurements that reflect the probability density distribution. With a large number of random pictures, the probability density distribution can be reconstructed with acceptable accuracy. In our instructional experiment with this metaphor, we found that most students could easily accept and interpret this type of probabilistic representation.¹²

In practice, we first help the students understand that the motion is periodic. Thus for the continuous case, we begin with the idea that the probability of finding the object in a small region Δx is proportional to Δt , the time that the object spends in Δx . When Δx is small and the velocity of the object does not change rapidly within Δx , Δt can be approximated by

$$\Delta t = \left| \frac{\Delta x}{v(x)} \right|, \quad (1)$$

where Δx represents a region defined by the interval (x_1, x_2) and equals $x_2 - x_1$. The quantity, $v(x)$, is the average velocity of the object in the interval (x_1, x_2) , where x is taken to be the center position of (x_1, x_2) . We use $P(x, \Delta x)$ to represent the probability for the bob to be in Δx and denote the period of the motion by T . Because the object will pass through the region twice in one period, the total time spent in Δx has a factor of 2. Then $P(x, \Delta x)$ can be obtained from

$$P(x, \Delta x) = 2 \frac{\Delta t(x, \Delta x)}{T} = \frac{2}{T} \left| \frac{\Delta x}{v(x)} \right|. \quad (2)$$

The first part of Eq. (2) is the core conceptual equation that allows students to make sense of the meaning of the probability. The second part of Eq. (2) provides a mechanism for calculating the result, using energy conservation to find $v(x)$. If we define $\rho(x)$ as the probability density, where $P(x, \Delta x) = \rho(x) \cdot \Delta x$, then $\rho(x)$ can be calculated from

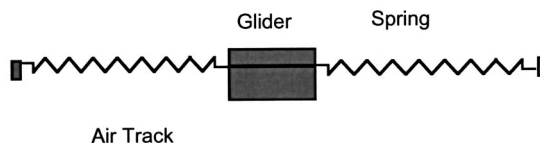


Fig. 2. A glider on an air track in harmonic oscillation.

$$\rho(x) = \lim_{\Delta x \rightarrow 0} \frac{P(x, \Delta x)}{|\Delta x|} = \frac{2}{T|v(x)|}. \quad (3)$$

The normalization condition can be written as

$$\oint \rho(x) dx = \int_0^T \frac{dt}{T} = 2 \int_0^{T/2} \frac{dt}{T} = \frac{2}{T} \int_0^L \frac{dx}{v} = 1. \quad (4)$$

At certain positions the velocity may become zero, making $\rho(x)$ go to infinity at that point. But typically the singularity is integrable and the probability in the small region about that point is finite. This problem can be a good exercise for advanced students.

With the random picture method, the probability $P(x, \Delta x)$ can be estimated by counting the number of pictures showing the object in Δx . Denote this number by $m(x, \Delta x)$ and let N represent the total number of pictures in the experiment. (It is necessary to have N large.) Then, the probability of finding an object in region Δx can be obtained from

$$P(x, \Delta x) = \lim_{N \rightarrow \infty} \frac{m(x, \Delta x)}{N}. \quad (5)$$

From the definition of the probability density, we obtain

$$\rho(x) = \lim_{\Delta x \rightarrow 0} \frac{P(x, \Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{m(x, \Delta x)}{N \Delta x}, \quad (6)$$

which also satisfies the normalization condition

$$\oint \rho(x) dx = \sum \frac{m(x, \Delta x)}{N} = \frac{N}{N} = 1. \quad (7)$$

It is assumed that different regions of Δx do not overlap.

Using digital video to find probability distribution—a pseudorandom method: Implementing a real experiment using the random picture method requires expensive hardware. In addition, the students need to learn how to handle the equipment, and the time required could be a large overhead distracting them from learning the real physics. An alternative way that we find suitable for lab and tutorial settings is to make a digital video of a working physical system in advance. Then in the class, the students can work on the digital videos with a *pseudorandom* method, picking random frames from the video as if they are taking random pictures of the real system. Here we discuss a simple example to show how this method works in practice.

The experiment is illustrated in Fig. 2. A glider on an air track is attached to two identical springs and is set to oscillate along the track. The motion of the glider is videotaped and digitized. Because the damping is small, we can obtain several complete cycles without noticeable changes in the amplitude of the oscillation.

From the video, we obtain a series of frames showing the position of the glider at different instants of time. Because the video is captured with a fixed rate of 30 frames per second (fps), the time interval between consecutive frames is a

Table I. Enumeration of frames for a complete period where $t_n = n \cdot \frac{1}{30}$; x_n is determined by the position of the cart in n th frame.

Frames (n)	t_n (s)	x_n
0	t_0	x_0
1	t_1	x_1
2	t_2	x_2
\vdots	\vdots	\vdots
59	t_{59}	x_{59}

constant equal to 1/30 s. The videos were made to include exactly one complete period of motion. Thus taking a picture at some random time t_r can be approximated by taking a picture at some time t in one period. This t can be calculated from

$$t = (t_r, \text{modulo } T). \tag{8}$$

In our experiment, the period of the oscillating glider is about 2 s, which gives a total of 60 frames. Each frame is labeled with a number n ($n=0-59$) and tagged by a time t_n , which represents the time relative to the beginning of the video. Then we can write

$$t_n = n \cdot \frac{1}{30}. \tag{9}$$

Next we construct a table containing a full set of frames in one complete period (see Table I). Suppose a student takes a picture at a random time t_r . One can use Eq. (8) to get t . Then t_n can be matched (from Table I) by finding a value closest to t . The video frame associated with the matched t_n is picked as the picture taken by the student at time t_r . This process is illustrated in Fig. 3. Obviously the outcome is not the real random picture that the student would get at t_r , but it is a reasonable approximation. Using a high-speed camera, one can increase the frame rate and improve the accuracy accordingly. For the glider experiment, the frame rate of 30 fps is enough for good results.

The position of the cart in each frame can be easily found with video analysis software such as *VideoPoint*TM. By choosing a large number of random frames ($N \sim 1000$), we can construct a data set for the positions of the cart at different random times. An Excel spreadsheet is developed using the internal Visual Basic functions to process the data.¹³ In the tutorial, the students work with *VideoPoint*TM to get the position of the cart in each frame and import the data to the Excel spreadsheet.

In the spreadsheet, the total range of the motion is divided into eight small regions with a fixed length of Δx , which has to be set larger than the maximum difference of positions of the glider between consecutive frames to make a more uni-

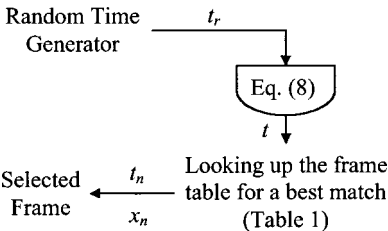


Fig. 3. The process of using the pseudo-random method to take random pictures.

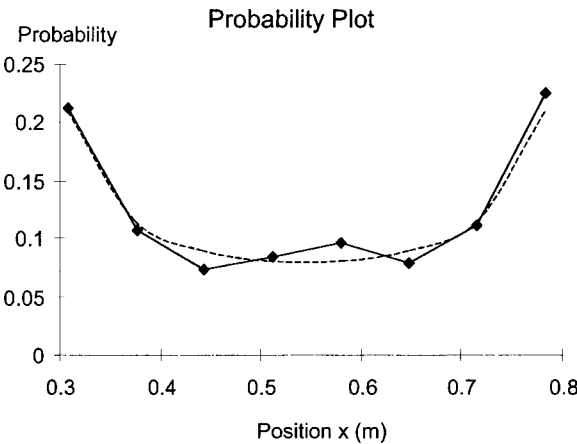


Fig. 4. The probability distribution of a glider in harmonic oscillation, created by a computer using the pseudo-random picture method. The plotted value represents the actual probability for the glider to be found in the each of the eight regions. The dashed line represents the theoretical curve.

form distribution and to avoid zero counts. After calculating the positions of the glider in all the pseudorandom pictures, the spreadsheet does a frequency count of the frames that have the position of the glider in each of the eight regions. The counted number is proportional to the probability of finding the glider inside the corresponding region and the probability density is obtained with Eq. (6). A typical plot of the calculated probability distribution is shown in Fig. 4. Smoother graphs can be obtained by using videos with a higher frame rate, which can reduce the error of the pseudo-random method and allows smaller values of Δx . Larger N can reduce the variance of the calculation.

For a harmonic oscillator, the analytical form of the probability density function can be easily found using Eq. (2) and energy conservation, which gives

$$\rho(x) = \frac{1}{\pi \sqrt{A^2 - x^2}}, \tag{10}$$

where A is the amplitude of the oscillation.

In the tutorial implemented in Physics 420, students are guided to derive Eq. (10) and compare it with the results obtained using the random picture idea. With the students in the Physics 263 class (spring 1996), the tutorial was simplified to focus on qualitative discussions of the random picture idea using the computer-generated results.

Tutorial activities: To help students develop a correct understanding, several experiments with simple one-

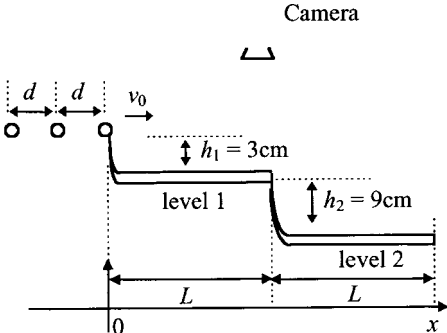


Fig. 5. An experiment with balls rolling on a stepped track.

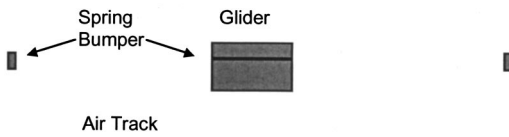


Fig. 6. An experiment showing a classical potential well.

dimensional systems were developed to use in a tutorial setting. The tutorial begins with systems of constant speed and progresses to more complicated systems such as the oscillating glider which has changing velocities. With these activities, students explore several key issues including the concept of probability density, relations between probability and probability density, mathematical formulation of probability density with simple classical systems, and normalization. In the following, we briefly describe two of the activities used in this tutorial.

(1) *Balls rolling down a stepped track.* A two-step track with sections of equal length is built as shown in Fig. 5. A series of balls with equal separation are set rolling towards the right with a very small initial velocity v_0 . The distance between the balls (denoted by d) is adjusted such that when a ball falls off the right edge of the track, the next ball enters the left side of the track.¹⁴ In this way only one ball is on the track at any time, thus creating a pseudoperiodic motion on the two lower segments of the track with a period, T , which equals the time that a ball takes to roll over the two lower steps. By choosing v_0 to be small, we can ignore the initial kinetic energy and simplify the calculation.

In the tutorial, we demonstrate the pseudoperiodic motion using a real setup and let the students play with it to get hands-on experience. The two equal steps of the track provide a straightforward example for the students to analyze the relation between probability and two different but constant velocities.

(2) *A classical potential well.* In the second experiment, we use the glider and the air track. This time, spring bumpers are attached to the glider and the two ends of the air track to produce elastic collisions at both ends (see Fig. 6). The potential energy of the glider is constant between the bumpers and rises quickly at the two ends like a deep square well.

IV. EVALUATION OF THE CURRICULUM

In the Physics 263 class of spring 1996, we implemented a tutorial that used the random picture metaphor with the clas-

Table III. Students responses on the question in final exam of Physics 263 class in spring 1996 (question C in Appendix).

Student response	Classical part	Quantum part
Students picked all correct choices	58%	69%
Students picked correct choices and also some incorrect choices	81%	84%
Students answered both parts with all correct answers	42%	

sical potential well and harmonic oscillator experiment. To see if the new instruction improved students understanding of probability, the Question B was given to the students in the class after they did the tutorial. The results from both the fall 1994 class and spring 1996 classes are shown in Table II. From the data, we can see that after the tutorial, 30% of the students used x as the spatial dimension to represent the position of the electron, whereas in the class without the tutorial only 9% of the students used the correct spatial dimension. The data also shows that 27% of the students could relate the probability of finding the electron in certain regions to the velocity of the electron. Although they are using a classical argument, we consider this result encouraging, compared to the situation of the class in fall 1994 where few could come up with any type of reasoning about probability. We also find 33% of the students in the spring 1996 class attempted to explain their reasoning and most of them used velocity and energy. In fall 1994, only 11% of the students attempted some kind of reasoning and few made any sense in terms of physics.

On the final exam for the class in the spring of 1996, we gave students a multiple-choice multiple-response (MCMR) question (Question C in the Appendix). Students responses on this question (Table III) also show encouraging results: 42% of the students could answer both the quantum and the classical part of the questions with no incorrect answers. Because it is a MCMR question, the number of students giving partially correct answers is much higher—around 80%. The results with the MCMR question suggest that the students who did not give perfect answers were in a mixed state, which is considered as a typical intermediate stage towards a favorable concept change.^{15–17}

In the Physics 420 class of spring 1998, the three activities

Table II. Physics 263 class students responses on conceptual quiz (question B in Appendix).

Types of student responses	Fall 94	Spring 96
Use energy levels/states (vertical dimension) to describe the position of an electron in a potential well (incorrect)	36%	27%
Use x (horizontal dimension) to describe the position of an electron in a potential well (correct)	9%	30%
Others	15%	14%
Blank	40%	29%
Implied student reasoning	Fall 94	Spring 96
Use velocity for reasoning of probability	0%	27%
Give reasoning (including correct and incorrect ones) ^a	11%	33%

^aThe reasoning of fall 94 students is mostly based on irrelevant issues. The reasoning of the spring 96 students is based on energy and velocity in a classical sense.

were integrated into two tutorials and students received more emphasis on the mathematical formulations of the probability density function in lecture. The class of fall 1998 was taught with traditional lectures only. For each of the two classes, we interviewed students after instruction (the class size is 15 to 20 students). The six students we interviewed in spring 1998 all used the random picture metaphor very fluently in their reasoning and could apply this idea to think about measurement of real physical systems. Three of them also gave a correct interpretation of quantum probability. In contrast, from the five students we interviewed in fall 1998, only one gave the correct quantum interpretation. The other four students failed to put together a reasonable mental picture for the probabilistic representation. Two of them could not give any reasoning at all; the other two students tried to provide some kind of reasoning, but failed to recognize certain crucial pieces such as the correct spatial dimension and the connection between quantum probability and the measurement of a real physical system.

V. CONCLUSION

It is well known that quantum physics has many difficult conceptual dualities waves and particles, position and momentum, the quantum character of small systems, and the classical limit. What is not always appreciated is that the teaching of quantum physics also contains instructional dualities that do not always appear in classical physics.

Quantum physics builds on a classical base, using many classical concepts and representations. If student understanding is weak in these areas, the learning of quantum physics may be difficult. However, strengthening this classical base can increase the likelihood that students will attempt to apply classical reasoning to quantum situations. For example, in Sec. IV, we discussed the results of interviews with students from a class that used tutorials on classical probability. We found that three of the six students were able to develop an appropriate understanding of quantum probability, but the remaining students used classical arguments in their reasoning and tried to associate the probability of finding an electron in a potential well with the velocity of the electron. On the other hand, among the five students we interviewed from the class without tutorials, four of them failed to provide any coherent explanation (not even a classical one).

We want our students to see physics as building a coherent and consistent representation of the physical world. Being exposed to quantum dualities can undermine student views that physics is consistent and makes sense. When we gave the Maryland Physics Expectations Survey to students in Physics 263 after they had four weeks of instruction on introductory quantum mechanics, we observed an unfavorable shift of students' views on the structure of their physics knowledge, where students appeared to view physics as a collection of isolated pieces rather than a coherent system of knowledge. Written comments indicated that quantum physics was the reason.

We want our students to learn to use mathematics as a representation of physics and to build their intuition and conceptual understanding into their equations. In quantum physics, the difficulty in building physical intuition tends to lead students to think that quantum physics is just math and lose the physical principles that lead us to choose the mathematics we use. In the Physics 263 final exam, about 1/4 of the students said that if a particular frequency failed to produce

photoelectrons, any change to the cathode would result in photoelectrons because if $eV_0 = hf - \phi$ gave zero before, changing ϕ will make it no longer zero. They focused on the math, failing to take into account the physical conditions (that the right-hand side must first be positive) which must be met before the equation can be applied.

We have discussed one issue necessary for the study of quantum physics: probability. There is an interaction with other issues such as reading potential energy diagrams and understanding and interpreting wave functions, which we have also studied but did not discuss here. Our research confirms that students often have difficulties in understanding basic issues of probability. In our calculus-based modern physics course, most students had never used a probabilistic representation to describe a physical system and they often held a strong deterministic view on physics phenomenon. To address these issues, we developed a random picture metaphor to help them build a mental bridge to the idea of probability and we developed tutorials using hands-on activities with classical systems. Our approach has helped, but represents only a first step. In classes with traditional instruction, most students were found to be confused by many of the basic ideas related to probability even after instruction. In such cases, students often misinterpret the wave function as the trajectory or the energy of the object. The students receiving tutorials developed a better understanding of issues related to probability and of those interviewed, most showed the ability to reason with and interpret probability densities. After instruction with tutorials, many students developed correct qualitative reasoning for probabilistic interpretations of classical systems and were able to use a correct understanding of probability density and the physical meaning of normalization.

ACKNOWLEDGMENTS

This investigation has been a collaborative effort by many members of the Physics Education Research Group at the University of Maryland. We particularly want to thank Richard Steinberg, Michael Wittmann, and Pratibha Jolly for discussions of these issues. Also greatly acknowledged are Professor Priscilla Laws for her assistance in developing some of the experiments and Professor Leonard E. Jossem for his help with this paper. This work was supported in part by the NSF grants DUE 965-2877, REC-0087788, REC-0126070, and the FIPSE grant P116B970186.

APPENDIX

Question A

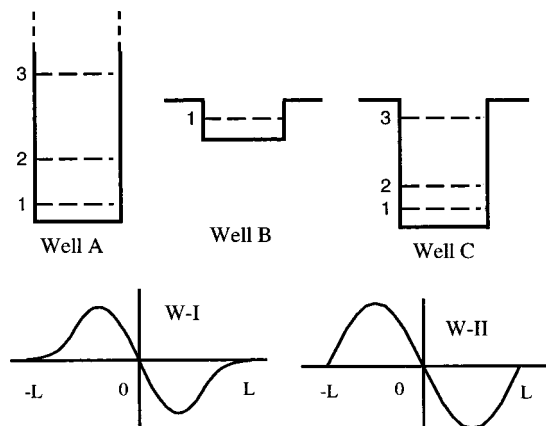
Conceptual quiz on probability

- (1) Consider the following coin tossing experiments:
Suppose you tossed a coin three times and get three heads in a row. Is the probability of getting a head on the next toss greater than, less than, or equal to 50%? Explain your reasoning.
If you toss a coin one hundred times, what do you expect to happen? If you toss it another one hundred times, do you expect to get the same number of heads and tails? Explain your reasoning.
- (2) Suppose the student average SAT score at Enormous State University is 1000. Your friend is in a writing class of 10 students. Her score was 1100. What is the most

probable average of the other 9 students? Explain your reasoning.

Question B

Conceptual quiz given in Physics 263 classes at University of Maryland in fall 1994 and spring 1996

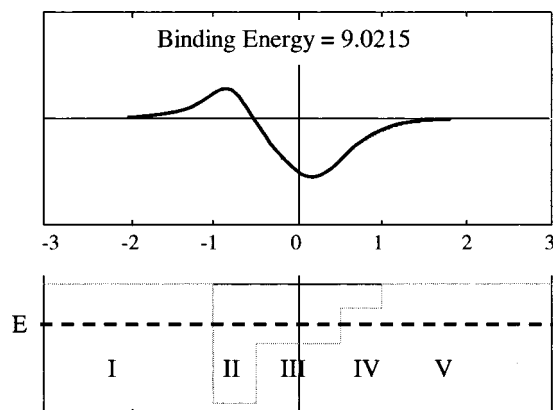


- (a) The wave functions shown are associated with two of the energy levels. Which ones do you think they are associated with? Explain your reasoning.
- (b) For these two states, approximately where in the well would the electron most likely be found? Explain your reasoning.

Question C

Exam question on quantum wave function given to Physics 263 class in fall 1996

The figure below is a plot of a one-dimensional potential energy function $U[x]$ (lower plot) and the wave function (upper plot) of an eigenstate for an electron in the influence of that potential. The energy, E , associated with that state is also shown on the energy plot. Distances are measured in nanometers and energies in eV.



The potential energy $U[x]$ is defined so that it has four values in five regions of the x axis:

Region I:	$x < -1$	$U[x] = 0$
Region II:	$-1 < x < -0.5$	$U[x] = -30$
Region III:	$-0.5 < x < 0.5$	$U[x] = -15$
Region IV:	$0.5 < x < 1$	$U[x] = -6$
Region V:	$1 < x$	$U[x] = 0$

(1.1) If the particle were moving classically (i.e., its motion were described by Newton's laws) in the potential $U[x]$,

and it had an energy E , which of the following statements would be true? List all that apply.

- (a) It would move the fastest when it was in region II.
- (b) It would move the fastest when it was in region III.
- (c) If we took a photograph of the particle at a random time, we would never find it in region IV.
- (d) If we took a photograph of the particle at a random time, we would be most likely to find it in region II.
- (e) If we took a photograph of the particle at a random time, we would be most likely to find it in region III.

(1.2) If the particle were moving quantum mechanically (i.e., its motion were described by the Schrödinger equation) in the potential $U[x]$, and it had an energy E , which of the following statements would be true? List all that apply.

- (a) If we measured the position of the particle at a random time, we would never find it in region I.
- (b) If we measured the position of the particle at a random time, we would never find it in region IV.
- (c) If we measured the position of the particle at a random time, we would most likely find it in region II.
- (d) If we measured the position of the particle at a random time, we would most likely find it in region III.
- (e) The state shown represents the lowest energy state that can be found in this well.

¹W. Casscells, A. Schoenberger, and T. Grayboys, "Interpretation by physicians of clinical laboratory results," *New England J. Med.* **299**, 999–1000 (1978); G. Keren, "Calibration and probability judgments; Conceptual and methodological issues," *Acta Psychol.* **77**, 217–273 (1991); L. A. Brenner, D. J. Koehler, V. Liberman, and A. Tversky, "Overconfidence in probability and frequency judgments: A critical examination," *Organ. Behav. Human Dec. Process* **65**, 212–219 (1996).

²M. Barnes, "Dealing with misconceptions about probability," *Australian Mathematics Teacher* **54**, 17–20 (1998); G. R. Fast, Using analogies to produce long term conceptual change: Overcoming high school mathematics students probability misconceptions, Meeting of the American Educational Research Association, Chicago, IL, March 24–28, 1997; P. Vahey, Toward an understanding of productive student conceptions of probability: The probability inquiry environment, Meeting of the American Educational Research Association, Chicago, IL, March 24–28, 1997; C. Konold, "Inconsistencies in students reasoning about probability," *J. Res. Math. Educ.* **24**, 392–414 (1993).

³The gambler's fallacy states that future results will compensate for previous (short term) results in order to bring things back to the average.

⁴J. Clement, "Using bridging analogies and anchoring intuitions to deal with students preconceptions in physics," *J. Res. Sci. Teach.* **30**, 1241–1257 (1993).

⁵L. Bao, Dynamics of student modeling: A theory, algorithms, and application to quantum mechanics, Ph.D. dissertation, University of Maryland, December 1999. Available on request from <http://www.physics.ohio-state.edu/lbao>

⁶Materials for this course are available from the Physics Education Research Group at the University of Maryland, <http://www.physics.umd.edu/qm/qmcourse/welcome.htm>

⁷L. C. McDermott and P.S. Shaffer, *Tutorials in Introductory Physics* (Prentice Hall, New York, 1998).

⁸This question was also used in a preinstructional survey where similar results (with less student explanations) were obtained. This makes us believe that the students inappropriate understanding was likely developed before instruction.

⁹The use of the vertical dimension as a measure of position rather than potential energy is associated with another student difficulty in understanding potential energy diagram. See Ref. 2 for more details.

¹⁰Modern experiments can actually trap and probe single quantum objects repeatedly, but this is not the norm.

¹¹J. P. Doering, J. H. Moore, and M. A. Coplan, "($e, 2e$) spectroscopy," *Rev. Mod. Phys.* **66**, 985–993 (1994).

¹²L. Bao, E. F. Redish, and R. S. Steinberg, "Student misunderstandings of the quantum wavefunction," *AAPT Announcer* **28**, 92 (1998).

¹³The current version of the software is available from the first author at <http://www.physics.ohio-state.edu/~lbao>

¹⁴In practice, the demonstrator simply drops a new ball onto the track as soon as one falls off.

¹⁵L. Bao and E. F. Redish (unpublished).

¹⁶R. K. Thornton, *Conceptual dynamics: Changing student views of force and motion*, Proceedings of the International Conference on Thinking Science for Teaching: the Case of Physics, Rome, 1994.

¹⁷D. P. Maloney and R. S. Siegler, "Conceptual competition in physics learning," *Int. J. Sci. Educ.* **15**, 283–295 (1993).

GORDON CONFERENCE ON PHYSICS RESEARCH AND EDUCATION: QUANTUM MECHANICS

The 2002 Gordon Conference on Physics Research and Education will focus on quantum mechanics and will be held on June 9–14, 2002 at Mount Holyoke College, South Hadley, Massachusetts. The goal of the conference is to bring together researchers who study and apply quantum mechanics, physics education researchers, and college and university level instructors of quantum mechanics for the purpose of promoting innovation in all aspects of teaching quantum mechanics throughout the undergraduate curriculum. The conference will include sessions and discussions about the desired content and outcome of courses, curriculum development using research on student understanding of topics in quantum mechanics, ways of approaching non-intuitive aspects of quantum theory, and the results of current research in physics that can be used to increase undergraduate student understanding of the concepts and applications of quantum mechanics. More information can be found at <http://www.grc.uri.edu/programs/2002/physres.htm>. Questions or suggestions about the Gordon Conference can be addressed to the organizers, Beth Ann Thacker (batcam@spudhammer.phys.ttu.edu), Harvey Leff (hsleff@csupomona.edu), or David Jackson (jacksond@dickinson.edu).