# Understanding probabilistic interpretations of physical systems: A prerequisite to learning quantum physics

Lei Bao

Department of Physics, The Ohio State University, 174 W 18<sup>th</sup> Ave., Columbus, OH 43210

Edward F. Redish Department of Physics, University of Maryland, College Park, MD 20742

# Abstract

Probability plays a critical role in making sense of quantum physics, but most science and engineering undergraduates have very little experience with the topic. A probabilistic interpretation of a physical system, even at a classical level, is often completely new to them, and the relevant fundamental concepts such as probability distribution and probability density are rarely understood. To address these difficulties and to help students build a model of how to think about probability in physical systems we developed a set of hands-on tutorial activities appropriate for use in a modern physics course for engineers. In this paper, we discuss some student difficulties with probability concepts and an instructional approach that uses a random picture metaphor and digital video technology.

# I. INTRODUCTION

A student's first course in quantum physics can be quite difficult. They have to think about phenomena for which they have no direct personal experience, they have to follow long chains of inference from experiment to what appear to be bizarre conclusions, and they have to deal with phenomena that fundamentally involve probabilities. This last introduces a number of difficulties. First, students of physics are rarely introduced to the use of probability in classical situations early in their studies, even in places where it would be appropriate, such as error analysis or statistical mechanics. Second, studies of people's understanding of probabilistic ideas in cognitive psychology<sup>1</sup> and mathematics education<sup>2</sup> research indicate that serious misunderstandings are common.

At the University of Maryland, the Physics Education Research Group carried out a project to explore the difficulties students had in learning quantum physics. The purpose of this paper is to highlight a few pieces of information from this research with emphasis on its practical values to instruction. Our research was carried out in two venues: the third semester of our introductory calculus-based engineering physics class (Physics 263) and an upper division one-semester course in quantum physics for engineers (Physics 420). Most of the emphasis was on this last venue and most of our curriculum development was tested there. The 263 class is required of all engineering majors. The 420 class is an upper division elective for engineers so it is considerably smaller (N  $\sim$  15-30). It is dominated by electrical engineers (80-90%) and is taught every semester.

After years of experience in a seemingly deterministic world, reinforced by learning classical physics, students can develop a strong deterministic view of the physical world. In most classical situations discussed in introductory classical physics classes, the behavior of a physical system can be precisely determined, and the emphasis is often on the construction of a detailed, quantified description of the motion of an object.

In quantum mechanics, students have to use and interpret probabilistic representations that are very different from the deterministic ones they have become accustomed to thinking of as "physics". In this paper, we first discuss our research, showing the kinds of difficulties students encounter with probability, including the presence of the gambler's fallacy<sup>3</sup> and the difficulty with the idea of a probability density. Then we consider instructional environments that can help students understand the fundamental concepts of probability and learn to use a probabilistic representation to interpret physical systems. To help students over the gaps, *bridges* are needed, and we make this bridging process in two steps.<sup>4</sup> The first is to help the students develop a basic understanding of probability using contexts that they are familiar with, e.g. classical systems. The second is to use new metaphors, hands-on analysis, and video tools to bring the students from an understanding of classical probability to an understanding of quantum probability.

#### **II. STUDENT DIFFICULTIES IN UNDERSTANDING PROBABILITY**

Many approaches in traditional instruction of quantum mechanics assume that classical pre-requisites such as the understanding of probability and energy diagrams are readily accessible to students. However, students often have a lot of difficulty with these pre-requisites.<sup>5</sup> Specifically, we want to see if the students were able to decipher the meaning of the phrase "probability of locating a particle in a certain region". In general, most undergraduate students are familiar only with a kinematical description of motion (of a particle in terms of a trajectory observed over a period of time). They may find it difficult to comprehend how a probabilistic representation relates to actual observations and how the measurement can be used to construct details of the particle behavior in the system.

Our observations were conducted with the students from two classes of the third semester of Physics 263 (one in the fall semester of 1994 and one in the spring semester of 1996) and two classes of Physics 420 (one each in the spring and fall semesters of 1998). The student population for both courses consisted mainly of science and engineering majors. In the Physics 263 course, only the lecture section taught by one of the authors (EFR) was studied. In the class of fall 1994 quantum physics was treated only through traditional lectures and in the class of spring 1996 we used three quantum tutorials, but none addressed the issue of probability.<sup>6</sup> The Physics 420 class was part of the curriculum development project, "A New Model Course in Applied Quantum Physics." For the three years of the project, the fall semester class was taught in a traditional fashion as a control by members of the department not participating in the research and in the spring semester class was taught in a modified fashion using our newly developed materials by members of the PERG. Most of the instructional innovation used a Tutorial format. Tutorials are a type of guided group-learning instruction developed by Lillian. C. McDermott and the physics education group at the University of Washington.<sup>7</sup> The quantum tutorials were developed by the University of Maryland PERG using a similar format.

The instruments used in this study to probe student thinking include two concept quizzes, one exam question, and student interviews. (The quiz and exam questions are given in the Appendix.) The set of problems in question A were designed to probe students' understanding of fundamental ideas in probability including independence of events and the gambler's fallacy. These problems were given at the beginning of the Physics 420 class to collect information on students' initial states. The question B was designed to probe a number of issues. First it tests to see whether the students understand the different shapes of the wavefunction of bound states. Second it tests to see if the students can make the link between the amplitude of the wavefunction and the probability density of a particle being in certain region. It also gives the information on student understandings of the potential well. This question was given to the two Physics 263 classes after instruction.

The problem shown in question C probes students' understanding of probabilistic interpretations with both classical systems and quantum systems. This question was used in the final exam of the Physics 263 class in spring 1996.

A total of 16 individual interviews were conducted with students from the Physics 263 and Physics 420 classes to investigate students' understanding of classical pre-requisites. The part of the interview relating to probability was based on the same issues as probed by the quiz problems but with a more open-ended style. In the following, we briefly summarize our observations.

#### • Predictability and the Stochastic Nature of Probability

In the five interviews conducted with the Physics 263 students (spring 1996) after instruction, we found that many of them (4/5) held a deterministic empirical intuition of probability. (These five students are all received a grade of "A" and are not representative of the overall population). Their descriptions show an incorrect understanding of the difference between the stochastic nature of any single observation and the somewhat determined expected distribution of the results of ensemble observations. These students bring with them the belief that small samples will replicate the probabilistic trends expected from a very large number of trials and that the specific result of any single measurement can be affected by the previous sequence of outcomes. For example, one of the students responded to the first part of question A with " … since you already have three heads in a row, you should have more chances to get a tail on the fourth time …".

A quantitative study with the students in the advanced class (Physics 420 with a total of 18 students) in the spring of 1998 also shows similar results. We used an open-ended survey that has a problem about coin-flipping experiment (see Appendix, question A.1). A majority of the students showed the gambler's fallacy: 61% thought that the result of a single coin-flipping event is dependent on the results of previous coin-flipping activity. In addition, 27% of the students thought that if the coin were flipped 100 times, there would be an exact 50/50 distribution for heads and tails. The last part in question A concerns probable values of students' SAT scores and also deals with the same issue. In this case more than 67% of the students considered that knowing one student's score would affect the probable average score of the other students.

#### • Understanding Probabilistic Representations

None of the students who visited during our office hours or participated in interviews reported having had any previous experience (before instruction) in using a probabilistic interpretation to think about a physical system. A very small number of them did recall the impression of doing some kind of math with probability in a math class but failed to remember any details of the mathematics. None had ever used probability to describe a real physical event.

In the physics 263 class of fall '94, we gave question B in the Appendix to the students on a quiz after instruction in quantum mechanics. On the part in which the students were given the wavefunction and asked to determine where in the well the electron would most likely be found, most of the students didn't use the correct spatial dimension, x, in their reasoning. The largest fraction, 40%, left the question blank; 36% of the class used the vertical dimension, V, as a spatial dimension for position.<sup>8</sup> (As suggested by interview results, many of these students appeared to consider that electrons with different energy states would also be in different places on the vertical dimension in the potential well.) Only 9% of the class used the correct dimension and among them only one student came up with the correct answer. Among all the students, only 11% provided some kind of reasoning for their answers on the quiz.

# **III. UNDERSTANDING PROBABILITY WITH CLASSICAL SYSTEMS**

The role of probability in microscopic systems is conceptually quite subtle. For most of the traditional experiments of quantum physics, it is not possible to set up an individual quantum object — e.g., an atom, molecule, or nucleus — and probe it repeatedly.<sup>9</sup> Instead, an ensemble of identically prepared objects is probed and the ensemble average is identified with the quantum average. Thus, in an (e,2e) experiment, thousands of electrons knock electrons out of thousands of different atoms or

molecules and for each individual case, the struck electrons' momenta before the collision is determined from momentum conservation. The result is interpreted as the probability distribution of finding momenta in a single atom or molecule.<sup>10</sup> Thus we note that *even if the fundamental mechanics of atoms and molecules were classical*, we would still need to describe most experiments with atoms using probabilities. This fact allows us to build a bridge through the use of probability in classical situations.

We introduce a metaphor, the *random picture* idea, as a fundamental tool for students to construct a probabilistic representation. Since atoms cannot be tracked or controlled individually, we ask students to consider a set of oscillating objects whose phases are random. We then ask the students to imagine taking a series of flash photographs of a single classically moving object at random times and using those photographs to predict where the object is most likely to be found. Based on this notion, hands-on activities and discussion questions were developed and used in tutorials where the students can apply this random picture metaphor to analyze real physical systems such as a cart moving back and forth on an air track.

### Building the probability density function

Consider a simple classical system with periodic motion such as a pendulum bob swinging back and forth. The traditional approach in classical mechanics is to think about the motion of the bob, the force on the bob during the motion, the velocity (or position) vs. time relation, etc. Such approaches often encourage the students to focus on the motion of the objects, which encourages a deterministic view of physical systems.

Another way to analyze this system is to think about its probabilistic aspects. For example, if one doesn't know when the motion of the bob started, its position at an arbitrary time is uncertain, but one can still predict the probability of finding the bob in certain regions, even though the exact position-time relation of the bob is unknown. Figure 1 is a time-exposure photograph of a white pendulum bob swinging against a black background. The brightness of a particular area is a relative measure of the amount of time that the bob spends in the corresponding region. It therefore reflects the distribution of the probability density for the bob to be found at different areas.



Figure 1. Time exposure photo of a white pendulum bob swinging against a black background.

The time-exposure photograph produces a continuous distribution function for the probability density. We can use the random picture idea to generate discrete measurements that reflect the probability density distribution. With a large number of "random pictures", the probability density distribution can be reconstructed with acceptable accuracy. In our instructional experiment with this metaphor, we found that most students can easily accept and interpret this type of probabilistic interpretation.<sup>11</sup>

The mathematical calculation is straightforward for the students in the calculus-based introductory physics courses. In practice, we first help the students understand that the motion is periodic. Thus for the continuous case, we begin with the idea that the probability of finding the object in a small region  $\Delta x$  is proportional to  $\Delta t$ , the time that the object spends in  $\Delta x$ . When  $\Delta x$  is small and the velocity of the object doesn't change rapidly within  $\Delta x$ ,  $\Delta t$  can be approximated by

$$\Delta t = \left| \frac{\Delta x}{v(x)} \right| \tag{1}$$

where  $\Delta x$  represents a region defined by the interval  $(x_1, x_2)$  and equals  $x_2 - x_1$ , and v(x) is the average velocity of the object inside the interval  $(x_1, x_2)$ , where x is taken to be the center position of  $(x_1, x_2)$ . Use  $P(x, \Delta x)$  to represent the probability for the bob to be in  $\Delta x$  and denote the period of the motion by T. Since the object will pass through the region twice in one period, the total time spent in  $\Delta x$  has a factor of 2. Then  $P(x, \Delta x)$  can be obtained from

$$P(x,\Delta x) = 2\frac{\Delta t(x,\Delta x)}{T} = \frac{2}{T} \left| \frac{\Delta x}{v(x)} \right|$$
(2)

The first part of Eq. (2) is the core conceptual equation that allows the students to make sense of the meaning of the probability. The second part of Eq. (2) provides a mechanism to calculate the result, using energy conservation to find v(x). Define  $\rho(x)$  as the probability density where  $P(x, \Delta x) = \rho(x) \cdot \Delta x$ . Then  $\rho(x)$  can be calculated from:

$$\rho(x) = \lim_{\Delta x \to 0} \frac{P(x, \Delta x)}{|\Delta x|} = \frac{2}{T|v(x)|}$$
(3)

The normalization condition can be written as

$$\oint \rho(x)dx = \int_{0}^{T} \frac{dt}{T} = 2\int_{0}^{T/2} \frac{dt}{T} = \frac{2}{T} \int_{0}^{L} \frac{dx}{v} = 1$$
(4)

It is worth mentioning that at certain positions the velocity may become zero, making  $\rho(x)$  go to infinity at that point. But typically the singularity is integrable and the probability in the small region around that point is finite. This problem can be a good exercise for advanced students.

With the random picture method, the probability  $P(x, \Delta x)$  can be estimated by counting the number of pictures showing the object in  $\Delta x$ . Denote this number with  $m(x, \Delta x)$  and let N represent the total number of pictures taken in an experiment. (It is necessary to have a large N.) Then, the probability of finding an object in region  $\Delta x$  can be obtained from

$$P(x,\Delta x) = \lim_{N \to \infty} \frac{m(x,\Delta x)}{N}$$
(5)

From the definition the probability density can be extracted by:

$$\rho(x) = \lim_{\Delta x \to 0} \frac{P(x, \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{m(x, \Delta x)}{N\Delta x},$$
(6)

which also satisfies the normalization condition:

$$\oint \rho(x)dx = \sum \frac{m(x,\Delta x)}{N} = \frac{N}{N} = 1$$
(7)

Here, it is assumed that different regions of  $\Delta x$  do not overlap.

#### Using digital video to find probability distribution – a pseudo-random method

Implementing a real experiment using the random picture method requires expensive hardware resources. In addition, the students need to learn how to handle the equipment, and the time required could be a large overhead distracting them from learning the real physics. An alternative way that we

find suitable for lab and tutorial settings is to make a digital video of a working physical system in advance. Then in the class, the students can work on the digital videos with a *pseudo-random* method, picking random frames from the video as if they are taking random pictures of the real system. Here we discuss a simple example to show how this method works in practice.

The experiment is illustrated in Figure 2. A glider on an air track is attached to two identical springs and is set to oscillate along the track. The motion of the glider is videotaped and digitized. Since the damping is small, we can get several complete cycles without noticeable changes in the amplitude of the oscillation.



Figure 2. A glider on an air track in harmonic motion

From the video, we can get a series of frames showing the position of the glider at different instants of time. Since the video is captured with a fixed rate of 30 frames per second (fps), the time interval between consecutive frames is a constant equal to  $\frac{1}{30}$  s. The videos used in our labs were made to include exactly one complete period of motion. Thus taking a picture at some random time  $t_r$  can be approximated by taking a picture at some time t inside one period. This t can be calculated from

$$t = (t_r \text{ modulo } T) \tag{8}$$

In the experiment, the period of the oscillating glider is about 2 seconds, which gives a total of 60 frames. Each frame is labeled with a number n ( $n = 0 \sim 59$ ) and tagged with a time  $t_n$ , which represents the time relative to the beginning of the video. Then we can write

$$t_n = n \cdot \frac{1}{30} \tag{9}$$

Now we build up a table containing a full set of frames in one complete period. (See Table 1.) Suppose a student takes a picture at a random time  $t_r$ . One can use Eq. (8) to get t. Then a  $t_n$  can be matched (from Table 1) by finding a value closest to t. The video frame associated with the matched  $t_n$ is picked as the picture taken by the student at time  $t_r$ . This process is illustrated in Figure 3. Obviously the outcome is not the real random picture that the student would get at  $t_r$ , but it is a reasonable approximation. Using a high-speed camera, one can increase the frame rate and improve the accuracy accordingly. For the glider experiment, the frame rate of 30 fps is enough for a good result.

Table 1.	<i>Frame table for a complete period where</i> $t_n = n \cdot \frac{1}{30}$	and $x_n$ is
	determined by the position of the cart in $n^{th}$ frame	

Frames (n)	$t_n(s)$	$x_n$
0	$t_0$	$x_0$
1	$t_1$	$x_1$
2	$t_2$	$x_2$
:	:	:
59	<i>t</i> <sub>59</sub>	<i>x</i> <sub>59</sub>

The position of the cart in each selected frame can be easily found with video analysis software such as *VideoPoint*<sup>TM</sup>. By picking a large number of random frames (N~1000), we can construct a data set for the positions of the cart at different random times. An Excel spreadsheet is developed using the internal

Visual Basic functions to process the data.<sup>12</sup> In the tutorial, the students work with  $VideoPoint^{TM}$  to get the position of the cart and import the data to the Excel spreadsheet. The spreadsheet follows students' commands to generate pseudo-random pictures and guide the students through the calculations.



Figure 3. The process of using the pseudo-random method to take random pictures

In the spreadsheet, the total range of the motion is divided into eight small regions with a fixed length of  $\Delta x$  (this has to be set larger than the maximum difference of positions of the glider between consecutive frames to make a more uniform distribution and to avoid zero counts). After calculating the positions of the glider in all the pseudo-random pictures, the spreadsheet does a frequency count of the frames that have the position of the glider in each of the eight regions. The counted number is proportional to the probability of finding the glider inside the corresponding region and probability density is obtained with Eq. (6). A typical plot of the calculated probability distribution is shown in Figure 4. Smoother graphs can be obtained when using videos with a higher frame rate, which can reduce the error of the pseudo-random method and allows smaller  $\Delta x$ . Larger N can reduce the variance of the calculation.



Figure 4. The probability distribution of a glider in harmonic motion, created by computer using the pseudo-random picture method. The plotted value represents the actual probability for the glider to be found in the each of the eight regions. The dashed line represents the theoretical curve.

For a harmonic oscillator, the analytical form of the probability density function can be easily found using Eq. (2) and energy conservation to be

$$\rho(x) = \frac{1}{\pi \sqrt{A^2 - x^2}}$$
(10)

where A is the amplitude of the oscillation.

These activities were used to develop Tutorial type instruction. In the Tutorial implemented in Physics 420, students are guided to derive this function and they can compare it with the results obtained with the computer using the random picture idea. With the students in the Physics 263 class (spring 1996), the Tutorial was simplified to focus on qualitative discussions of the random picture idea using the computer-generated results.

#### **Tutorial activities**

To help students develop correct understanding, a number of experiments with simple onedimensional systems were developed to use in a Tutorial setting. The Tutorial begins with systems of constant speed and progresses to complicated systems such as the oscillating glider that has changing velocities. With these activities, students explore several key issues including the concept of probability density, relations between probability and probability density, mathematical formulation of probability density with simple classical systems, and the concept of normalization. In the following, we briefly describe two specific activities used in this tutorial.

#### 1. Balls rolling down a stepped track.

A two-step track with sections of equal length is built as shown in Figure 5. A series of balls with equal separation are set rolling towards the right with a very small initial velocity  $v_0$ . The distance between the balls (denoted with *d*) is adjusted such that when a ball falls off the right edge of the track, the next ball enters the left side of the track.<sup>13</sup> This ensures that only one ball is on the track at any time and thus creates a pseudo-periodic motion on the two lower segments of the track with a period, *T*, that equals the time that a ball takes to roll over the two lower steps. By setting a small  $v_0$ , we can ignore the initial kinetic energy and simplify the calculation.

In the tutorial, we demonstrate the pseudo-periodic situation using a real setup and let the students play with it to get hands-on experience. The two equal steps of the track provide a straightforward example for the students to analyze the relation between probability and two different but constant velocities.



Figure 5. An experiment with balls rolling on a stepped track

## 2. A classical potential well.

In the second experiment, we use the glider and the air track. This time, spring bumpers are attached to the glider and the two ends of the air track to produce elastic collisions at both ends (see Figure 6). The potential energy of the glider is constant between the bumpers and rises quickly at the two ends like a deep square well.



Figure 6. An experiment showing a classical potential square well

# **IV. EVALUATION OF THE CURRICULUM**

In the Physics 263 class of spring 1996, we implemented a tutorial that used the random picture metaphor with the classical potential well and harmonic oscillator experiment. To see if the new instruction improved students' understanding of probability, the quiz question (question B) was given to the students in the spring '96 class after they did the tutorial. The results from both fall '94 class and spring '96 classes are shown in Table 2. From the data, we can see that after the tutorial, 30% of the students used *x* as the spatial dimension to represent the position of the electron whereas in the class without the tutorial only 9% of the students used the correct spatial dimension. The data also shows that a significant fraction, 27%, of the students could relate the probability of finding the electron in certain regions to the velocity of the electron. Although they are using a classical argument, we consider this result encouraging, compared to the situation of the class in fall 1994 where few could come up with any type of reasoning about probability. We also find 33% of the students in the spring 1996 class attempted to explain their reasoning and most of them used velocity and energy. In fall 1994, only 11% of the students attempted some kind of reasoning and few of those made any sense in terms of physics.

Types of Student Responses	Fall 94	Spring 96
Use energy levels/ states (vertical dimension) to describe the position of an electron in a potential well (incorrect)	36%	27%
Use x (horizontal dimension) to describe the position of an electron in a potential well (correct)	9%	30%
Others	15%	14%
Blank	40%	29%
Implied Student Reasoning	Fall 94	Spring 96
Use velocity for reasoning of probability	0%	27%
Give reasoning (including correct and incorrect ones)*	11%	33%

Table 2. Physics 263 class students' responses on conceptual quiz (question B in Appendix)

<sup>\*</sup>The reasoning of fall 94 students is mostly based on irrelevant issues. The reasoning of the spring 96 students is based on energy and velocity in a classical sense.

On the final exam for the class in the spring of 1996, we gave students a multiple-choice multipleresponse (MCMR) question (question C shown in the Appendix). Students' responses on this question (Table 3) also show encouraging results: 42% of the students can answer both the quantum and the classical part of the questions with all correct choices (i.e. no incorrect answers). Since it is a MCMR question, the number of students giving partially correct answers is much higher – around 80%. This suggests that the students who didn't give perfect answers were in a mixed state, which is considered as a typical intermediate stage towards a favorable concept change.<sup>14, 15, 16</sup>

Student Response	<b>Classical Part</b>	Quantum Part
Students picked all correct choices	58%	69%
Students picked correct choices and also some incorrect choices	81%	84%
Students answer both parts with all correct answers	42%	

Table 3. Students' responses on the question in final exam of Physics 263 class in spring 1996(question C in Appendix)

In the Physics 420 class of spring 1998, the three activities were integrated into two tutorials and students received more emphasis on the mathematical formulations of the probability density function in lecture. The Physics 420 class in fall 1998 was taught with traditional lectures only. For each of the two classes, we interviewed about half a dozen students after instruction (the class size is 15~20 students). The six students we interviewed in spring 1998 all used the random picture metaphor very fluently in their reasoning and could apply this idea to think about measurement of real physical systems. Three of them also gave a correct interpretation of quantum probability. In contrast, from the five students we interviewed in fall 1998, only one gave the correct quantum interpretation. The other four students failed to put together a reasonable mental picture for the probabilistic representation. Two of them couldn't give any reasoning at all; the other two students tried to provide some kind of reasoning but failed to recognize certain crucial pieces such as the correct spatial dimension and the connection between quantum probability and the measurement of a real physical system.

#### **V. CONCLUSION**

It is well known that quantum physics has many difficult conceptual "dualities" — waves and particles, position and momentum, the quantum character of small systems and the classical limit. What is not always appreciated is that the teaching of quantum physics also contains instructional dualities that do not always appear in classical physics.

1. Quantum physics builds on a classical base, using many classical concepts, variables, and representations. If students are weak on these items, learning of quantum physics may be difficult. However, strengthening that classical base can increase the likelihood that students will attempt to apply classical reasoning to quantum situations. For example, in the previous section, we discussed the results of six interviews with students from a class that used tutorials on classical probability, where we found that half of the six students were able to develop an appropriate understanding of quantum probability, however, the remaining students used classical arguments in their reasoning – students tried to associated the probability of finding an electron in a potential well with the velocity of the electron and were able to build a classical interpretation. On the other hand, among the five students we interviewed from the class without tutorials, four of them failed to provide any coherent explanation (not even a classical one).

2. We want our students to see physics as building a coherent and consistent representation of the physical world. "Quantum thinking" requires the ability to use models that appear contradictory in a coherent way. Being exposed to quantum dualities can undermine student views that physics is consistent and can "make sense." When we gave the Maryland Physics Expectations Survey (MPEX)<sup>17</sup> to students following the quantum section of Physics 263, we observed a sharp drop in students' expectations on the coherence variable. Written comments indicated that quantum physics was the reason.

3. We want our students to learn to use mathematics as a representation of physics and to build their intuitions and conceptual understanding into their equations. In quantum physics, the difficulty in building physical intuition tends to lead students to think that quantum physics "is just math" and lose the physical principles that lead us to choose the mathematics we use. In the Physics 263 final exam, about <sup>1</sup>/<sub>4</sub> of the students said that if a particular frequency failed to produce photoelectrons, any change to the cathode would result in photoelectrons since "if  $eV_0 = hf - \phi$  gave zero before, changing  $\phi$  will make it no longer zero." They focused on the math, failing to take into account the physical conditions (that the right side must first be positive) that must be met before the equation can be applied.

In this paper we have discussed one narrow issue needed for the study of quantum physics: probability. There is an interaction with other issues such as reading potential energy diagrams and understanding and interpreting wave functions, which we have also studied but that we do not discuss here. On this narrow issue of probability, our research confirms that students often have difficulties in understanding basic issues of probability. In our calculus-based modern physics course, most students had never used a probabilistic representation to describe a physical system and they often held a strong deterministic view on physics phenomenon. To address these issues, we developed a random picture metaphor to help them build a mental bridge to the idea of probability and we developed Tutorials using hands-on activities with classical systems.

Our approach helped, but only represents a first step. In classes with traditional instruction, most students were found to be much confused by many of the basic ideas related to probability even after instruction. In such cases, students often misinterpret the wavefunction as the trajectory or the energy of the object. The students receiving tutorials developed a better understanding on issues related to probability and of those interviewed, most showed the ability to reason with and interpret probability densities. After instruction with Tutorials, many students developed correct qualitative reasoning for probabilistic interpretations of classical systems and were able to use a correct understanding of probability density and the physical meaning of normalization.

## ACKNOWLEDGMENT

This investigation has been a collaborative effort by many members of the Physics Education Research Group at the University of Maryland. We particularly want to thank Richard Steinberg, Michael Wittmann, and Pratibha Jolly for discussions of these issues. Also greatly acknowledged are Professor Priscilla Laws for her assistance in developing some of the experiments and Professor Leonard E. Jossem for his help with this manuscript. This work is supported in part by the NSF grants DUE 965-2877, REC-0087788 and the FIPSE grant P116B970186.

# **Endnotes and Reference:**

- <sup>1</sup> W. Casscells, A. Schoenberger, and T. Grayboys, "Interpretation by physicians of clinical laboratory results," *New England J. of Medicine* **299**, 999-1000 (1978); G. Keren, "Calibration and probability judgments; Conceptual and methodological issues," *Acta Psychologica* **77**, 217-273 (1991); L. A. Brenner, D. J. Koehler, V. Liberman, and A, Tversky, "Overconfidence in probability and frequency judgments: A critical examination," *Organizational Behavior and Human Decision Process* **65**(3), 212-219 (1996).
- <sup>2</sup> M. Barnes, "Dealing with Misconceptions about Probability," *Australian Mathematics Teacher* v54 n1 p17-20 Mar (1998); G. R. Fast, "Using Analogies To Produce Long Term Conceptual Change: Overcoming High School Mathematics Students' Probability Misconceptions," Paper presented at the Annual Meeting of the American Educational Research Association (Chicago, IL, March 24-28, 1997). 16 p., March (1997); P. Vahey,

"Toward an Understanding of Productive Student Conceptions of Probability: The Probability Inquiry Environment," Paper presented at the Annual Meeting of the American Educational Research Association (Chicago, IL, March 24-28, 1997). 17 p., March (1997); Konold, C. "Inconsistencies in Students' Reasoning about Probability," *Journal for Research in Mathematics Education*. v24 n5 p392-414 Nov (1993).

- <sup>3</sup> The gambler's fallacy states that future results will compensate for previous (short term) results in order to "bring things back to the average." Note that this is applied opposite to the Bayesian approach. Thus, "10 heads in a row" is interpreted as evidence that tails will start to appear instead of as evidence for a biased coin.
- <sup>4</sup> J. Clement, "Using bridging analogies and anchoring intuitions to deal with students" preconceptions in physics," J. Res. Sci. Teach. **30**:10, 1241-1257 (1993).
- <sup>5</sup> L. Bao, "Dynamics of Student Modeling: A Theory, Algorithms, and Application to Quantum Mechanics," Ph.D. dissertation, University of Maryland, December 1999. Available on request from http://www.physics.ohio-state.edu/~lbao.
- <sup>6</sup> Materials for this course are available from the website of the Physics Education Research Group at the University of Maryland, http://www.physics.umd.edu/qm/qmcourse/welcome.htm.
- <sup>7</sup> L. C. McDermott, P.S. Shaffer, *Tutorials in Introductory Physics* (Prentice Hall, New York, 1998)
- <sup>8</sup> This is associated with their difficulty in understanding potential energy diagram. See reference 2 for more details.
- <sup>9</sup> Modern experiments can actually trap and probe single quantum objects repeatedly, but this is not the norm.
- <sup>10</sup> J. P. Doering, J.H. Moore, and M. A. Coplan, "(e, 2e) Spectroscopy," Rev. Mod. Phys., 66, (1994) 985-993.
- <sup>11</sup> L., Bao, E.F. Redish, and R.S. Steinberg, "Student Misunderstandings of the Quantum Wavefunction," AAPT Announcer 28 (2), 92 July (1998).
- <sup>12</sup> The current version of the software is available from the first author at his web site, http://www.physics.ohio-state.edu/~lbao.
- <sup>13</sup> In practice, the demonstrator simply drops a new ball onto the track as soon as one falls off.
- <sup>14</sup> L. Bao and E. F. Redish, "Model Analysis: Assessing the Dynamics of Student Learning," submitted to *Cognition and Instruction*.
- <sup>15</sup> R. K. Thornton, "Conceptual Dynamics: Changing Student Views of Force and Motion," Proceedings of the International Conference on Thinking Science for Teaching: the Case of Physics. Rome, Sept. 1994.
- <sup>16</sup> D. P. Maloney and R. S. Siegler, "Conceptual competition in physics learning," *Int. J. Sci. Educ.*, **15** (3), 283-295, (1993).
- <sup>17</sup> E. F. Redish, R. N. Steinberg, and J. M. Saul, "Student expectations in introductory physics," Am. J. Phys. 66, 212-224 (1998)