Functional Identities for QED

Lagrangian for QED with gauge-fixing term

$$\mathcal{L} = \mathcal{L}_{QED} + \mathcal{L}_{gf}$$

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi} \gamma^\mu D_\mu \Psi - m \bar{\Psi} \Psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\mu \Psi = \partial_\mu \Psi - ie A_\mu \Psi$$

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2$$

equations of motion:

$$(i \gamma^\mu \partial_\mu - m) \Psi = -e A_\mu \gamma^\mu \Psi$$

$$\Box^{\mu\nu} A_\nu = -e \bar{\Psi} \gamma^\mu \Psi$$

$$\text{where } \Box^{\mu\nu} = g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu + \frac{1}{\xi} \partial^\mu \partial^\nu$$

current conservation

$$\partial_\mu (\bar{\Psi} \gamma^\mu \Psi) = 0$$

complete photon propagator

$$\langle \Omega | T A_\mu(x) A_\nu(y) | \Omega \rangle = \langle A_\mu(x) A_\nu(y) \rangle$$

$$= \frac{\int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp(iS) A_\mu(x) A_\nu(y)}{\int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp(iS)}$$

path integral weighted by photon field

$$\int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp(iS) A_\nu(y)$$

is invariant under shifting field $A_\mu(x)$ by a function

$$A_\mu(x) \to A_\mu(x) + \alpha_\mu(x)$$

1st order changes:

$$\delta A_\nu(y) = \alpha_\nu(y)$$

$$\delta S = \delta \int d^4x \left(\tfrac{1}{2} A_\mu \Box^{\mu\nu} A_\nu + e A_\mu \bar{\psi} \gamma^\mu \psi \right)$$

$$= \int d^4x \left(\tfrac{1}{2} \alpha_\mu \Box^{\mu\nu} A_\nu + \tfrac{1}{2} A_\mu \Box^{\mu\nu} \alpha_\nu + e \alpha_\mu \bar{\psi} \gamma^\mu \psi \right)$$

$$= \int d^4x \, \alpha_\mu(x) \left(\Box^{\mu\nu} A_\nu + e \bar{\psi} \gamma^\mu \psi \right)$$

1st order changes must be 0:

$$0 = \int \mathcal{D}A\,\mathcal{D}\bar\Psi\,\mathcal{D}\Psi \exp(iS)\Big[\,a_\nu(y)$$

$$+ A_\nu(y)\, i\int d^4x\; \alpha_\mu(x)\big(\square^{\mu\nu} A_\nu(x) + e\bar\Psi\gamma^\mu\Psi(x)\big)\Big]$$

express as an integral with factor $\alpha_\mu(x)$:

$$0 = i\int d^4x\, \alpha_\mu(x) \int \mathcal{D}A\,\mathcal{D}\bar\Psi\,\mathcal{D}\Psi \exp(iS)\Big[-ig^\mu{}_\nu\, \delta^4(x-y)$$

$$+ A_\nu(y)\big(\square^{\mu\lambda} A_\lambda(x) + e\bar\Psi\gamma^\mu\Psi(x)\big)\Big]$$

must hold for all functions $\alpha_\mu(x)$

$\Longrightarrow$ coefficient of $\alpha_\mu(x)$ must be 0

divide by unweighted path integral integral

$$\boxed{\square_x^{\mu\lambda}\langle A_\lambda(x) A_\nu(y)\rangle + e\langle \bar\Psi\gamma^\mu\Psi(x) A_\nu(y)\rangle - i g^\mu{}_\nu\, \delta^4(x-y) = 0}$$

$$\square^{\mu\nu} = g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu + \tfrac{1}{\xi}\partial^\mu\partial^\nu$$

Schwinger-Dyson equation for photon field

Ward identity for photon field

path integral weighted by photon field

$$\int \mathcal{D}A \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, \exp(iS)\, A_\nu(y)$$

invariant under gauge transformation

$$\Psi(x) \longrightarrow e^{+i\alpha(x)}\,\Psi(x)$$
$$\bar{\Psi}(x) \longrightarrow e^{-i\alpha(x)}\,\bar{\Psi}(x)$$
$$A_\mu(x) \longrightarrow A_\mu(x) - \tfrac{1}{e}\partial_\mu \alpha(x)$$

- change in path integral at 1st order in α
- express in the form $\int d^4x\, \alpha(x)\, F(x)$
- must be zero for all $\alpha(x) \Longrightarrow F(x) = 0$
- divide by unweighted path integral

$$\boxed{\frac{1}{\xi}\,\Box_x \partial_x^\mu \langle A_\mu(x) A_\nu(y)\rangle = i\frac{\partial}{\partial x^\nu}\delta^4(x-y)}$$

Fourier transform in both variables

by applying $\int d^4x\, e^{iq\cdot x} \int d^4y\, e^{ik\cdot y}$

$$\int d^4x\, e^{iq\cdot x} \int d^4y\, e^{ik\cdot y}\, \frac{1}{\xi} \Box_x \partial_x^\mu \langle A_\mu(x) A_\nu(y) \rangle$$

$$= \int d^4x\, e^{iq\cdot x} \int d^4y\, e^{ik\cdot y}\, i \frac{\partial}{\partial x^\nu} \delta^4(x-y)$$

integrate by part

$$\frac{1}{\xi}\left(i q^2 q^\mu\right) \int d^4x\, e^{iq\cdot x} \int d^4y\, e^{ik\cdot y} \langle A_\mu(x) A_\nu(y) \rangle$$

$$= -i q_\nu \int d^4x\, e^{iq\cdot x} \int d^4y\, e^{ik\cdot y}\, \delta^4(x-y)$$

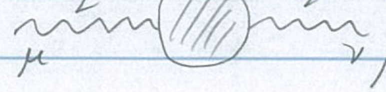
$$\frac{1}{\xi}\left(i q^2 q^\mu\right) \langle \tilde{A}_\mu(q) \tilde{A}_\nu(k) \rangle = -i q_\nu (2\pi)^4 \delta^4(q+k)$$

momentum-space Green function: $\langle \tilde{A}_\mu(q) \tilde{A}_\nu(k) \rangle = D_{\mu\nu}(q) (2\pi)^4 \delta^4(k+q)$

factor out δ-function

$$\frac{1}{\xi} q^2 q^\mu D_{\mu\nu}(q) = -i q_\nu$$

diagrammatic equation

$\frac{1}{\xi} q^2 q^\mu \Big($  $\Big) = -i q_\nu$

complete electron propagator

$$\langle \Omega | T\, \psi_i(x)\, \bar{\psi}_j(y) | \Omega \rangle \equiv \langle \psi_i(x)\, \bar{\psi}_j(y) \rangle$$

$$= \frac{\int \mathcal{D}A\, \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \exp(iS)\, \psi_i(x)\, \bar{\psi}_j(y)}{\int \mathcal{D}A\, \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \exp(iS)}$$

path integral weighted by electron field

$$\int \mathcal{D}A \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \exp(iS)\, \psi_i(y)\, \bar{\psi}_j(z)$$

invariant under local phase transformations

$$\psi(x) \longrightarrow e^{+i\alpha(x)}\psi(x)$$
$$\bar{\psi}(x) \longrightarrow e^{-i\alpha(x)}\bar{\psi}(x)$$
$$A_\mu(x) \longrightarrow A_\mu(x)$$

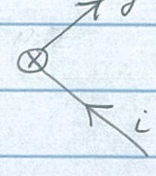
- change in path integral at 1st order in α
- express in the form $\int d^4x\, \alpha(x)\, F(x)$
- must be zero for all $\alpha(x) \implies F(x) = 0$
- divide by unweighted path integral

Ward-Takahashi identity

$$\boxed{\begin{aligned} &i\frac{\partial}{\partial x^\mu}\langle \bar{\psi}\gamma^\mu\psi(x)\, \psi_i(y)\, \bar{\psi}_j(z)\rangle \\ &= \delta^4(x-z)\langle \psi_i(y)\, \bar{\psi}_j(x)\rangle - \delta^4(x-y)\langle \psi_i(x)\, \bar{\psi}_j(z)\rangle \end{aligned}}$$

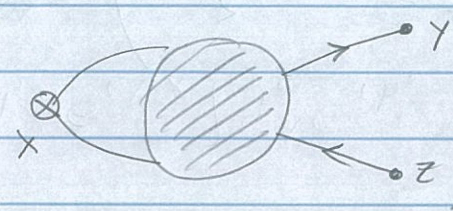
diagrammatic representation

vertex for current $J^\mu = \bar{\psi}\gamma^\mu\psi$:



$= (\gamma^\mu)_{ji}$

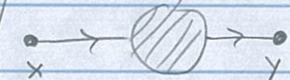
$i\frac{\partial}{\partial x^\mu}$ (



)

$= \delta^4(x-y)$ (

) $- \delta^4(x-z)$ (



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Fourier transform in all variables:

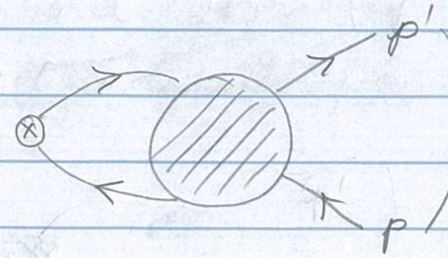
$$\int d^4x\, e^{-iq\cdot x} \int d^4y\, e^{ip'\cdot y} \int d^4z\, e^{-ip\cdot z}$$

$$q_\mu \langle \tilde{J}^\mu(-q)\, \tilde{\psi}_i(p')\, \tilde{\bar{\psi}}_j(-p) \rangle$$

$$= \langle \tilde{\psi}_i(p'-q)\, \tilde{\bar{\psi}}_j(-p) \rangle - \langle \tilde{\psi}_i(p')\, \tilde{\bar{\psi}}_j(-p-q) \rangle$$

factor out $(2\pi)^4\delta^4(p'-p-q)$
to get identity for momentum-space Green function

q_μ (



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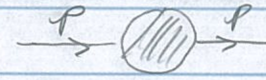

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