**Z⁰ Production in Photon Collisions**

The cross section for Z⁰ production from the reaction \( e^+(p_1)e^-(p_2) \rightarrow Z^0 \) is

\[
\sigma[e^+(p_1)e^-(p_2) \rightarrow Z^0] = \frac{12\pi B_{ee} \Gamma_Z^2 s}{M_Z^2 \left[ (s-M_Z^2)^2 + M_Z^2 \Gamma_Z^2 \right]},
\]

where \( s = (p_1 + p_2)^2 \) is the square of the center-of-mass energy and \( B_{ee} \) is the branching fraction of Z⁰ into e⁺e⁻.

A. The parton distribution for electrons in a photon that undergoes a collision involving a large momentum scale \( Q \gg m_e \) is

\[
f_{e^-/\gamma}(x, Q) = \frac{\alpha}{2\pi} \log \frac{Q}{m_e} \left[ x^2 + (1-x)^2 \right].
\]

Use the charge conjugation symmetry of QED to deduce the parton distribution \( f_{e^+/\gamma}(x, Q) \) for positrons in a photon.

\[
f_{e^+/\gamma}(x, Q) = f_{e^-/\gamma}(1-x, \frac{Q}{m_e}) = \frac{\alpha}{2\pi} \mathcal{L}_{\nu\gamma} \left[ x^2 + (1-x)^2 \right] \]

B. The Z⁰ can be produced in a positron-photon collision through the splitting of the photon into a collinear e⁺e⁻ pair. Draw the Feynman diagram for the reaction \( e^+(p_1)\gamma(q_2) \rightarrow e^+Z^0(P) \) with a virtual electron with momentum \( xq_2 \) collinear to that of the incoming photon. Label the momenta of all the lines.

C. Express the cross section for \( e^+(p_1)\gamma(q_2) \rightarrow e^+Z^0 \) as an integral over the longitudinal momentum fraction \( x \) of electrons in the photon.

\[
\sigma\left[ e^+(p_1)\gamma(q_2) \rightarrow e^+Z^0 \right] = \frac{1}{M_Z^2} \int_x \sigma_{e^-/\gamma}(x, M_Z) \sigma\left[ e^+(p_1) e^-(xq_2) \rightarrow Z^0 \right]
\]

D. Express the parton center-of-mass energy \( \hat{s} \) for the e⁺e⁻ collision in terms of the center-of-mass energy \( s \) for the e⁺\( \gamma \) collision. Express \( \hat{s} \) in terms of \( M_Z \).

\[
\hat{s} = (p_1 \times q_2)^2 = 2 \times p_1 \cdot q_2 = (p_1 \times p_2) = \lambda \Delta
\]
E. The $Z^0$ can be produced in a photon-photon collision through the splitting of both photons into collinear $e^+e^-$ pairs. Draw the two Feynman diagrams for the reaction $\gamma(q_1)\gamma(q_2) \rightarrow e^+e^-Z^0(P)$ in which there is a virtual $e^+$ collinear to one photon with momentum fraction $x_+$ and a virtual $e^-$ collinear to the other photon with momentum fraction $x_-$. Label the momenta of all the lines.

F. Express the cross section for $\gamma(q_1)\gamma(q_2) \rightarrow e^+e^-Z^0$ as a double integral over momentum fractions $x_+$ and $x_-$. (There are two terms in the cross section.)

$$J^+[J^+\gamma(q_1)\gamma(q_2) \rightarrow e^+e^-Z^0] = \int_0^1 dx_+ \int_0^{1-x_+} dx_- \left[ \left( e^+(x\xi_1)e^-(x\xi_2) \rightarrow Z^0 \right) + \left( e^+(x\xi_2)e^-(x\xi_1) \rightarrow Z^0 \right) \right]$$

G. Express the parton center-of-mass energy $\hat{s}$ for the $e^+e^-$ collision in terms of the center-of-mass energy $s$ for the $\gamma\gamma$ collision. Express $\hat{s}$ in terms of $M_Z$.

$$\hat{s} = (x_+\xi_1 + x_-\xi_2)^2 = 2x_+x_-\xi_1\xi_2 \approx x_+x_- (\xi_1 + \xi_2)^2 = x_+x_- \hat{\Delta}$$

H. Calculate the integral $\int_0^1 dx f_{e^-/\gamma}(x, Q)$, which gives the mean number of collinear electrons in a photon that undergoes a collision with a large momentum scale $Q$.

$$\int_0^1 dx f_{e^-/\gamma}(x, Q) = \frac{\alpha}{2\pi} \ln \frac{Q}{m_e} \int_0^1 dx \left[ x^2 + (1-x)^2 \right] = \frac{3}{2} \cdot \frac{\alpha}{2\pi} \ln \frac{Q}{m_e}$$

I. The parton distributions for a photon satisfy the momentum sum rule

$$\int_0^1 dx \left[ f_{\gamma/\gamma}(x, Q) + f_{e^-/\gamma}(x, Q) + f_{e^+/\gamma}(x, Q) \right] = 1.$$