Renormalization of Yukawa Model

The (incomplete) Lagrangian for the Yukawa model is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - g\phi\bar{\psi}\gamma_5\psi.$$

The Feynman rule for the vertex is $-ig\gamma_5$. The Feynman rules for the counterterms are

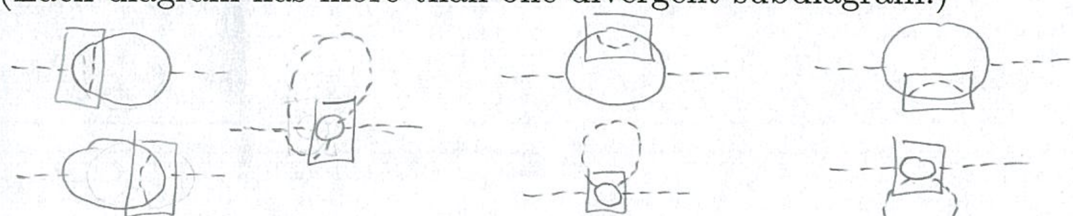
$$-i\,\delta g\gamma_5, \qquad -i\left[\delta M + \delta Z_f\,(\not{p} - M)\right], \qquad -i\left[\delta m^2 + \delta Z_b\,(p^2 - m^2)\right].$$

A. Draw the three 2-loop diagram for the boson self-energy $-i\,\Pi(p^2)$. Determine the superficial degree of divergence D of one of the diagrams by power counting the loop momenta.

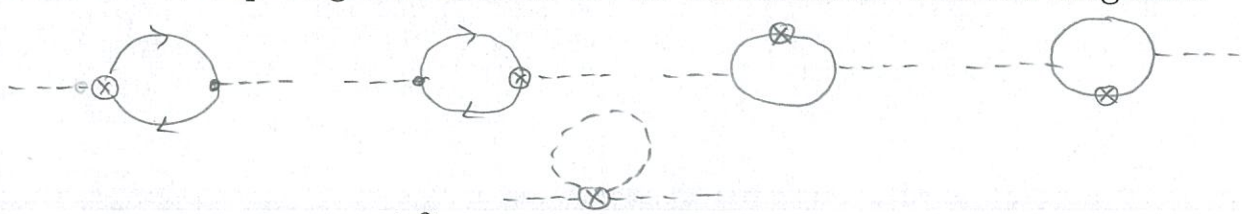
$$\int (d^4k)^2 \frac{1}{k^2}\left(\frac{1}{k}\right)^4 = \int d^8k \frac{1}{k^6}$$

$$\Longrightarrow D = 2$$

B. Identify the UV divergent subdiagrams of each of the 2-loop diagrams. (Each diagram has more than one divergent subdiagram.)



C. Draw the 1-loop counterterm diagrams for the boson self-energy $-i\,\Pi(p^2)$ that would be required to cancel the ultraviolet divergences from subdiagrams of the two 2-loop diagrams if there are no cancellations between diagrams.



The boson self-energy $\Pi(p^2)$ can be expanded around a negative invariant mass: $p^2 = -\mu^2$. The expansion in powers of $p^2 + \mu^2$ can expressed in the form

$$\Pi(p^2) = \Pi_1(-\mu^2) + \Pi'(-\mu^2)\,(p^2 + \mu^2) + \Pi_{\text{remainder}}(p^2).$$

D. Identify the superficial degree of divergence of each term. Verify that the divergent terms are polynomial in the 4-momentum p^μ.

$\Pi_1(-\mu^2)$: $D = 2$ $\quad \Pi'(-\mu^2)$: $D = 0$ $\quad \Pi_{\text{remainder}}(p^2)$: $D \le -2$ (finite)

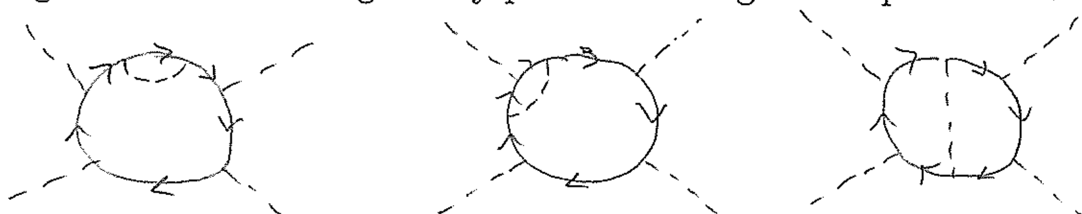
The 1PI 4-boson Green function $G(p_i^2, p_i{\cdot}p_j)$ with incoming momenta p_1, p_2, p_3, and p_4 is a function of 6 independent scalar variables: p_1^2, p_2^2, p_3^2, p_4^2, $p_1{\cdot}p_2$, $p_1{\cdot}p_3$, and $p_2{\cdot}p_3$, with the constraint

$$(p_1+p_2)^2+(p_1+p_3)^2+(p_2+p_3)^2=p_1^2+p_2^2+p_3^2+p_4^2.$$

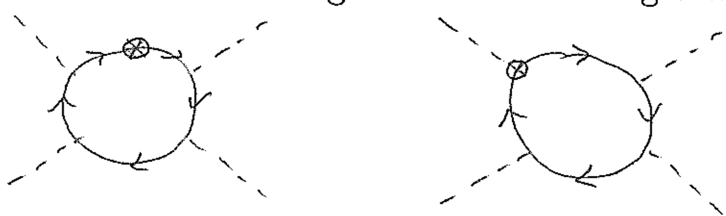
E. Verify that the symmetric Euclidean point $p_1^2=p_2^2=p_3^2=p_4^2=-\mu^2$ and $p_1{\cdot}p_2=p_1{\cdot}p_3=p_2{\cdot}p_3=\frac{1}{3}\mu^2$ is compatible with the constraint.

$$3\left[(-\mu^2)+2\cdot\tfrac{1}{3}\mu^2+(-\mu^2)\right] = -4\mu^2 \qquad 4(-\mu^2) = -4\mu^2$$

F. Draw three 2-loop diagram for $-i\,G(p_i^2, p_i{\cdot}p_j)$ that do not differ by permutations of any external legs. Determine the superficial degree of divergence D for one diagram by power counting the loop momenta.



G. Draw the one-loop counterterm diagrams for $-i\,G(p_i^2, p_i{\cdot}p_j)$ diagram that cancel ultraviolet divergences from subdiagrams of the 2-loop diagrams.



The 1PI 4-boson Green function $G_1(p_i^2, p_i{\cdot}p_j)$ can be expanded around the symmetric Euclidean point. The expansion in powers of $p_i^2+\mu^2$ and $p_i{\cdot}p_j-\frac{1}{3}\mu^2$ can expressed in the form

$$G(p_i^2, p_i{\cdot}p_j) = G(p_i^2=-\mu^2, p_i{\cdot}p_j=\tfrac{1}{3}\mu^2) + G_{\text{remainder}}(p_i^2, p_i{\cdot}p_j).$$

H. Identify the superficial degree of divergence of each term. Verify that the divergent terms are (a trivial) polynomial in the 4-momenta p_i^μ.

$D=0$ $\qquad\qquad$ $D=-2$ (finite)

I. The UV divergence can be cancelled by a counterterm $-i\,\delta\lambda$. Write down the interaction term in the Lagrangian that would give such a counterterm.

$$\mathcal{L}_{\text{counterterm}} = -\frac{\delta\lambda}{4!}\phi^4$$