Path Integral for Harmonic Oscillator

The path integral for the harmonic oscillator with angular frequency ω on the time interval from 0 to T is

$$\int \mathcal{D}x \ \exp\left(iS[x]/\hbar\right) = \int \mathcal{D}x \ \exp\left(\frac{i}{\hbar}\int_0^T dt \left[\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2\right]\right),$$

where the boundary conditions on the path $x(t)$ are $x(0) = 0$ and $x(T) = 0$.

1. Two possible $i\epsilon$ prescriptions on the frequency ω are

$$\omega^2 \longrightarrow \omega^2 \pm i\varepsilon \quad \text{with } \varepsilon \to 0^+.$$

Which prescription will improve the convergence of the path integral by suppressing the contributions from regions of large $x(t)$?

$\omega^2 \longrightarrow \omega^2 - i\epsilon$ gives a convergence factor

$$\exp\left(\frac{i}{\hbar}\int_0^T dt\left[(-\tfrac{1}{2})(-i\epsilon)x^2\right]\right) = \exp\left(-\frac{\epsilon}{\hbar}\int_0^T dt\, \tfrac{1}{2}x^2\right)$$

2. Use integration by parts to express the action as $S[x] = \int_0^T dt\left[\frac{1}{2}x\mathcal{O}x\right]$, where the differential operator is $\mathcal{O} = -\partial_t^2 - \omega^2$.

$$S[x] = \int_0^T dt\left[\frac{d}{dt}\left(\tfrac{1}{2}x\dot{x}\right) - \tfrac{1}{2}x\ddot{x} - \tfrac{1}{2}\omega^2 x^2\right]$$

$$= \int_0^T dt\left[\tfrac{1}{2}x\left(-\left(\tfrac{d}{dt}\right)^2 - \omega^2\right)x\right] + \tfrac{1}{2}x\dot{x}\Big|_0^T$$

$$= \int_0^T dt\left[\tfrac{1}{2}x\,\mathcal{O}\,x\right] + 0$$

The Gaussian path integral can be evaluated as

$$\int \mathcal{D}x \ \exp\left(\frac{i}{\hbar}\int_0^T dt\left[\tfrac{1}{2}x\mathcal{O}x\right]\right) = \mathcal{N}\ (\mathrm{Det}\,\mathcal{O})^{-1/2},$$

where $\mathcal{N}$ is a (divergent) normalization factor that depends on T but not on ω.

3. Replace $\mathcal{O}$ in this integration formula by the unit operator $\mathbb{1}$ to get an expression for $\mathcal{N}$ as a Gaussian integral.

$$\mathcal{N} = \int \mathcal{D}x \ \exp\left(\frac{i}{\hbar}\int_0^T dt\left[\tfrac{1}{2}x^2\right]\right)$$

The eigenvalue problem for the differential operator $\mathcal{O} = -\partial_t^2 - \omega^2$ is

$$(-\partial_t^2 - \omega^2)x = \lambda x,$$
$$x(0) = x(T) = 0.$$

4. Solve the eigenvalue equation for $\mathcal{O}$ by finding all its eigenvalues λ_n and its eigenfunctions $x_n(t)$. (Hint: consider functions of the form $\sin(n\pi t/T)$.)

$$(-\partial_t^2 - \omega^2)\sin(n\pi t/T) = \left(\frac{n^2\pi^2}{T^2} - \omega^2\right)\sin(n\pi t/T)$$

$\sin(n\pi t/T) = 0$ at $t=0$ and $t=T$

eigenfunctions: $x_n(t) = \sin(n\pi t/T)$ eigenvalues: $\lambda_n = \frac{n^2\pi^2}{T^2} - \omega^2$

5. Express the functional determinant $\mathrm{Det}\,\mathcal{O}$ as an infinite product over all positive integers n.

$$\mathrm{Det}\,\mathcal{O} = \prod_{n=1}^{\infty}\left(\frac{n^2\pi^2}{T^2} - \omega^2\right)$$

6. Euler's infinite product formula for the sine function is

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}\left(1 - z^2/n^2\right).$$

Use this formula to determine the dependence of $\mathrm{Det}\,\mathcal{O}$ on ω.

$$\mathrm{Det}\,\mathcal{O} = \prod_{n=1}^{\infty}\left[\frac{n^2\pi^2}{T^2}\left(1 - \frac{\omega^2T^2}{n^2\pi^2}\right)\right]$$

$$= \left(\prod_{n=1}^{\infty}\frac{n^2\pi^2}{T^2}\right)\prod_{n=1}^{\infty}\left(1 - \frac{\omega^2T^2}{n^2\pi^2}\right) = \left(\prod_{n=1}^{\infty}\frac{n^2\pi^2}{T^2}\right)\frac{\sin(\omega T)}{\omega T}$$

7. Express the ω-independent factor of $\mathrm{Det}\,\mathcal{O}$ as the exponential of an expression involving the divergent sums $\sum_{n=1}^{\infty} 1$ and $\sum_{n=1}^{\infty}\log(n)$.

$$\prod_{n=1}^{\infty}\frac{n^2\pi^2}{T^2} = \exp\left(\sum_{n=1}^{\infty}\log\frac{n^2\pi^2}{T^2}\right) = \exp\left(2\log\frac{\pi}{T}\sum_{n=1}^{\infty}1 + 2\sum_{n=1}^{\infty}\log n\right)$$

8. Give a final result for the path integral with explicit dependence on ω.

$$\int \mathcal{D}x\, e^{iS[x]/\hbar} = \left(\prod_{n=1}^{\infty}\frac{n^2\pi^2}{T^2}\right)\int\mathcal{D}x\, e^{\frac{i}{\hbar}\int_0^T dt\,\frac{1}{2}\dot{x}^2}\cdot\frac{\sin(\omega T)}{\omega T}$$