Path Integral for Driven Harmonic Oscillator

The path integral for a harmonic oscillator with angular frequency ω and driving force $f(t)$ on the time interval $0 < t < T$ is a functional of f:

$$Z[f] = \int \mathcal{D}x \, \exp\left(\frac{i}{\hbar}\int_0^T dt \left[\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 + fx\right]\right).$$

The integral is over paths $x(t)$ that satisfy the initial and final boundary conditions $x(0) = 0$ and $x(T) = 0$.

1. The phase factor is $iS[x]/\hbar$, where S is the action:

$$S[x] = \int_0^T dt \left[\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 + f(t)x\right].$$

An infinitesimal variation $\delta x(t)$ of the path gives an infinitesimal variation δS of the action. Use integration by parts to express δS as the sum of an integral with a factor of $\delta x(t)$ in the integrand and endpoint contributions from $t = 0$ and $t = T$.

$$\delta S = \int_0^T dt \left[\tfrac{1}{2} 2\dot{x}\delta\dot{x} - \tfrac{1}{2}\omega^2 2x\delta x + f\delta x\right]$$

$$= \int_0^T dt \left[\frac{d}{dt}(\dot{x}\delta x) - \ddot{x}\delta x - \omega^2 x\delta x + f\delta x\right]$$

$$= \dot{x}\delta x\Big|_0^T + \int_0^T dt \left(-\ddot{x} - \omega^2 x + f\right)\delta x$$

2. The action for paths $x(t)$ that satisfy boundary conditions $x(0) = 0$ and $x(T) = 0$ can be expressed as

$$S[x] = \int_0^T dt \left[\tfrac{1}{2}x\mathcal{O}x + fx\right] = \int_0^T dt \left[\tfrac{1}{2}(x + f\mathcal{O}^{-1})\mathcal{O}(x + \mathcal{O}^{-1}f) - \tfrac{1}{2}f\mathcal{O}^{-1}f\right],$$

where $\mathcal{O} = -\partial_t^2 - \omega^2$. Verify that the two expressions for the integral are equivalent.

$$\int_0^T dt \left[\tfrac{1}{2}x\mathcal{O}x + \tfrac{1}{2}f\mathcal{O}^{-1}\mathcal{O}x + \tfrac{1}{2}f\mathcal{O}^{-1}\mathcal{O}x + \tfrac{1}{2}f\mathcal{O}^{-1}\mathcal{O}\mathcal{O}^{-1}f - \tfrac{1}{2}f\mathcal{O}^{-1}f\right]$$

$$= \int_0^T dt \left[\tfrac{1}{2}x\mathcal{O}x + \tfrac{1}{2}fx + \tfrac{1}{2}fx + \tfrac{1}{2}f\mathcal{O}^{-1}f - \tfrac{1}{2}f\mathcal{O}^{-1}f\right]$$

$$= \int_0^T dt \left[\tfrac{1}{2}x\mathcal{O}x + fx\right]$$

If there is no driving force ($f(t) = 0$ for all t), the path integral can be evaluated using the Gaussian integration formula:

$$\int \mathcal{D}x \, \exp\left(\frac{i}{\hbar}\int_0^T dt \, \left[\tfrac{1}{2}x\mathcal{O}x\right]\right) = \mathcal{N}\left(\operatorname{Det}\mathcal{O}\right)^{-1/2},$$

where $\mathcal{N}$ is a (divergent) normalization factor that depends on T.

3. Evaluate the path integral for $Z[f]$ analytically as a functional of f by making the shift $x(t) \to x(t) - \mathcal{O}^{-1}f(t)$ in the integration variable and then evaluating the resulting Gaussian integral over x.

$$\int \mathcal{D}x \; exp\left(i\int_0^T dt\left(\tfrac{1}{2}(x+O^{-1}f)O(x+O^{-1}f) - \tfrac{1}{2}fO^{-1}f\right)\right)$$

$$x \to x - O^{-1}f$$

$$= \int \mathcal{D}x \; exp\left(i\int_0^T dt\left(\tfrac{1}{2}xOx - \tfrac{1}{2}fO^{-1}f\right)\right)$$

$$= \int \mathcal{D}x \; exp\left(i\int_0^T dt\left[\tfrac{1}{2}xOx\right]\right) \times exp\left(-i\int_0^T dt\left[\tfrac{1}{2}fO^{-1}f\right]\right)$$

$$= \mathcal{N}\,(det\; O)^{-\frac{1}{2}} \; exp\left(-i\int_0^T dt\left[\tfrac{1}{2}fO^{-1}f\right]\right.$$

The result for the path integral can be expressed as

$$Z[f] = Z[0]\exp\left(-\frac{i}{\hbar}\int_0^T dt \int_0^T dt' \left[\tfrac{1}{2}f(t)\mathcal{O}^{-1}(t,t')f(t')\right]\right).$$

4. Calculate the first variational derivative of Z:

$$\frac{\delta}{\delta f(t_1)}Z[f] = Z[f]\left(-\frac{i}{\hbar}\int_0^T dt' \,\tfrac{1}{2}O^{-1}(t_1,t')f(t') - \frac{i}{\hbar}\int_0^T dt\,\tfrac{1}{2}f(t)O^{-1}(t,t_1)\right)$$

$$= Z[f]\left(-\frac{i}{\hbar}\int_0^T dt\; O^{-1}(t_1,t)f(t)\right)$$

5. Calculate the second variational derivative of Z and evaluate it at $f = 0$:

$$\frac{\delta}{\delta f(t_2)}\frac{\delta}{\delta f(t_1)}Z[f] = Z[f]\left[\left(-\frac{i}{\hbar}\int_0^T dt\; O^{-1}(t_1,t)f(t)\right)\left(-\frac{i}{\hbar}\int_0^T dt'\, O^{-1}(t_2,t')f(t')\right) - \frac{i}{\hbar}O^{-1}(t_1,t_2)\right]$$

$$\frac{\delta}{\delta f(t_2)}\frac{\delta}{\delta f(t_1)}Z[f=0] = Z[0]\left(-\frac{i}{\hbar}O^{-1}(t_1,t_2)\right)$$