Path Integral for Two-Level Quantum System

The quantum mechanics of a two-level system with energy splitting ω_0 on the time interval $0 < t < T$ can be formulated in terms of a path integral over a complex Grassman function $\psi(t)$ in the presence of a time-dependent complex Grassmann source $\eta(t)$:

$$Z[\eta] = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp\left(iS\right), \quad S = \int_0^T dt\left[\tfrac{i}{2}(\psi^\dagger \partial_t \psi - \partial_t \psi^\dagger \psi) - \omega_0 \psi^\dagger \psi + \eta^\dagger \psi + \psi^\dagger \eta\right],$$

where the integral is over paths $\psi(t)$ that satisfy $\psi(0) = 0$ and $\psi(T) = 0$.

A. Use integration by parts to express the action for $\eta = 0$ in the form

$$\textstyle\int_0^T dt\left[\frac{i}{2}(\psi^\dagger \partial_t \psi - \partial_t \psi^\dagger \psi) - \omega_0 \psi^\dagger \psi\right] = \int_0^T dt\left[\psi^\dagger \mathcal{O} \psi\right],$$

where $\mathcal{O}$ is a differential operator.

$$\int_0^T dt\left[\tfrac{i}{2}(\psi^\dagger\partial_t\psi - \partial_t\psi^\dagger\psi) - \omega_0\psi^\dagger\psi\right] = \int_0^T\left[\tfrac{i}{2}\psi^\dagger\partial_t\psi - \tfrac{i}{2}\left(\partial_t(\psi^\dagger\psi) - \psi^\dagger\partial_t\psi\right) - \omega_0\psi^\dagger\psi\right]$$

$$= -\tfrac{i}{2}\psi^\dagger\psi\Big|_0^T + \int_0^T dt\left[i\psi^\dagger\partial_t\psi - \omega_0\psi^\dagger\psi\right] = 0 + \int_0^T dt\, \psi^\dagger(i\partial_t - \omega_0)\psi$$

B. Verify that the action can be expressed as

$$\textstyle\int dt\left[\psi^\dagger \mathcal{O} \psi + \eta^\dagger \psi + \psi^\dagger \eta\right] = \int dt\left[(\psi + \mathcal{O}^{-1}\eta)^\dagger \mathcal{O}(\psi + \mathcal{O}^{-1}\eta) - \eta^\dagger \mathcal{O}^{-1}\eta\right].$$

$$= \int dt\left[\psi^\dagger O\psi + (O^{-1}\eta)^\dagger O\psi + \psi^\dagger O O^{-1}\eta + (O^{-1}\eta)^\dagger O O^{-1}\eta - \eta^\dagger O^{-1}\eta\right]$$

$$= \int dt\left[\psi^\dagger O\psi + \eta^\dagger O^{-1} O\psi + \psi^\dagger\eta + \cancel{\eta^\dagger O^{-1}\eta} - \cancel{\eta^\dagger O^{-1}\eta}\right]$$

$$= \int dt\left[\psi^\dagger O\psi + \eta^\dagger\psi + \psi^\dagger\eta\right]$$

The Gaussian path integral over a complex Grassmann function $\psi(t)$ is

$$\int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp\left(i\textstyle\int_0^T dt\left[\psi^\dagger \mathcal{M} \psi\right]\right) = \mathcal{N}\, \mathrm{Det}\mathcal{M}.$$

where $\mathcal{N}$ is a divergent factor that depends on T.

C. Use a shift in the integration path $\psi(t)$ to express $Z[\eta]$ as the product of a functional of $\eta(t)$ and a factor that does not depend on η.

$$Z[\eta] = \int D\psi^\dagger D\psi \exp\left(i\int dt\left[(\psi + O^{-1}\eta)^\dagger O(\psi + O^{-1}\eta) - \eta^\dagger O^{-1}\eta\right]\right) \qquad \psi \to \psi - O^{-1}\eta$$

$$= \int D\psi^\dagger D\psi \exp\left(i\int dt\left[\psi^\dagger O\psi - \eta^\dagger O^{-1}\eta\right]\right)$$

$$= \int D\psi^\dagger D\psi \exp\left(i\int dt\left[\psi^\dagger O\psi\right]\right) \times \exp\left(-i\int dt\, \eta^\dagger O^{-1}\eta\right)$$

The path integral for the two-level quantum system in the presence of the Grassmann source $\eta(t)$ can be expressed as

$$Z[\eta] = Z[0] \exp\Big(- i \textstyle\int dt \int dt' \left[\eta^\dagger(t) \mathcal{O}^{-1}(t,t') \eta(t')\right]\Big).$$

D. Calculate the variational derivative of $Z[\eta]$ with respect to η, expressing it as the product of $Z[\eta]$ and a Grassmann function.

$$\frac{\delta}{\delta\eta(t_1)} Z[\eta] = Z[\eta]\left(-i\,(-1) \int dt\, \eta^\dagger(t)\, \mathcal{O}^{-1}(t,t_1)\right)$$

$$= Z[\eta]\left(+i \int dt\, \eta^\dagger(t)\, \mathcal{O}^{-1}(t,t_1)\right)$$

E. Calculate the second variational derivative of $Z[\eta]$ with respect to η and $\eta^\dagger$.

$$\frac{\delta}{\delta\eta^\dagger(t_2)} \frac{\delta}{\delta\eta(t_1)} Z[\eta] = Z[\eta]\Big[\left(-i \int dt'\, \mathcal{O}^{-1}(t_2,t')\, \eta(t')\right)\left(+i \int dt\, \eta^\dagger(t)\, \mathcal{O}^{-1}(t,t_1)\right)$$

$$+ i\, \mathcal{O}^{-1}(t_2,t_1)\Big]$$

The propagator for the two-level quantum system can be expressed in terms of variational derivatives of a path integral:

$$\langle 0|T\psi(t_2)\psi^\dagger(t_1)|0\rangle = \frac{1}{Z[\eta]}\left(-i\frac{\delta}{\delta\eta^\dagger(t_2)}\right)\left(i\frac{\delta}{\delta\eta(t_1)}\right) Z[\eta]\Big|_{\eta=0}.$$

F. Evaluate the propagator using the result of part E.

$$\langle 0|T\psi(t_2)\psi^\dagger(t_1)|0\rangle = i\, \mathcal{O}^{-1}(t_2,t_1)$$

G. The operator $\mathcal{O}^{-1}$ can be expressed as an integral transform:

$$\mathcal{O}^{-1}(t,t') = \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, \frac{1}{\omega - \omega_0 + i\epsilon}.$$

Verify this by showing that $\mathcal{O}\,\mathcal{O}^{-1}(t,t') = \delta(t-t')$.

$$(i\partial_t - \omega_0)\, \mathcal{O}^{-1}(t,t') = \int \frac{d\omega}{2\pi} (\omega - \omega_0)\, e^{-i\omega(t-t')} \frac{1}{\omega - \omega_0 + i\epsilon} = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} = \delta(t-t')$$