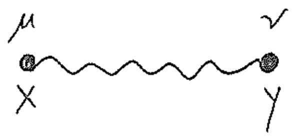
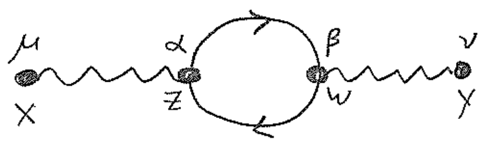
Ward Identity for Photon Propagator in Momentum Space

For QED with a covariant gauge-fixing term, the Ward identity for the complete photon propagator in position space is

$$\partial_x^\mu \partial_x^2 \langle A_\mu(x) A_\nu(y) \rangle = i\xi \tfrac{\partial}{\partial x^\nu} \delta^4(x-y).$$

A. Draw the Feynman diagrams for the complete photon propagator at order e^0 and at order e^2. Label each vertex with a position and a Lorentz index.





B. The photon propagator at order e^0 is

$$\langle A_\mu(x) A_\nu(y) \rangle = \int \frac{d^4q}{(2\pi)^4} e^{iq.(x-y)} \frac{i\big[-g_{\mu\nu} + (1-\xi) q_\mu q_\nu / q^2\big]}{q^2 + i\epsilon}.$$

Verify the Ward identity at order e^0.

$$\partial_x^\mu \partial_x^2 \langle A_\mu(x) A_\nu(y) \rangle = \int \frac{d^4q}{(2\pi)^4} \left(-i q^\mu q^2 e^{iq\cdot(x-y)}\right) \frac{i[-g_{\mu\nu} + (1-\xi) q_\mu q_\nu / q^2]}{q^2 + i\epsilon}$$

$$= \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot(x-y)} q^\mu [-g_{\mu\nu} + (1-\xi) q_\mu q_\nu / q^2] = -\xi \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot(x-y)} q_\nu$$

$$= i\xi \frac{\partial}{\partial x^\nu} \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot(x-y)}$$

The Fourier transform of the photon field is $\tilde{A}_\mu(q) = \int d^4x\, e^{iq.x} A_\mu(x)$. The Fourier transform of the Ward identity in both variables is

$$\int d^4x\, e^{iq.x} \int d^4y\, e^{ik.y}\, \partial_x^\mu \partial_x^2 \langle A_\mu(x) A_\nu(y) \rangle = \int d^4x\, e^{iq.x} \int d^4y\, e^{ik.y}\, i\xi \tfrac{\partial}{\partial x^\nu} \delta^4(x-y).$$

C. Use integration by parts to express this as an equation for the correlation function $\langle \tilde{A}_\mu(q) \tilde{A}_\nu(k) \rangle$ of Fourier-transformed photon fields.

$$\int d^4x \left(-\partial^\mu \partial^2 e^{iq\cdot x}\right) \int d^4y\, e^{ik\cdot y} \langle A_\mu(x) A_\nu(y) \rangle = \int d^4x \left(-i\xi \partial_\nu e^{iq\cdot x}\right) \int d^4y\, e^{ik\cdot y} \delta^4(x-y)$$

$$i q^\mu q^2 \int d^4x\, e^{iq\cdot x} \int d^4y\, e^{ik\cdot y} \langle A_\mu(x) A_\nu(y) \rangle = \xi q_\nu \int d^4x\, e^{iq\cdot x} e^{ik\cdot x}$$

$$i q^2 q^\mu \langle \tilde{A}_\mu(q) \tilde{A}_\nu(k) \rangle = \xi q_\nu (2\pi)^4 \delta^4(k+q)$$

D. The complete photon propagator $D_{\mu\nu}(q)$ in momentum space is obtained from $\langle \tilde{A}_\mu(q) \tilde{A}_\nu(k) \rangle$ by factoring out a delta function:

$$\langle \tilde{A}_\mu(q) \tilde{A}_\nu(k) \rangle = D_{\mu\nu}(q)\, (2\pi)^4 \delta^4(q+k).$$

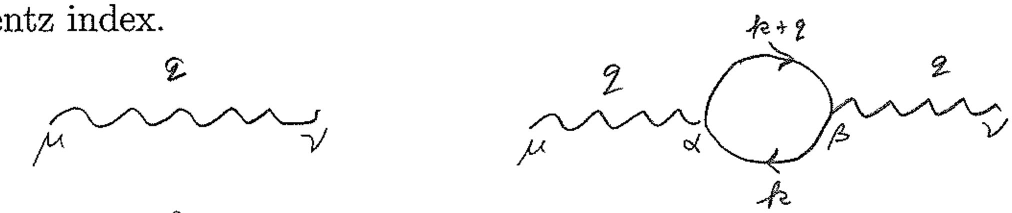
Write down the Ward identity for $D_{\mu\nu}(q)$.

$$i q^2 q^\mu D_{\mu\nu}(q) = \xi q_\nu$$

The Ward identity for the complete photon propagator in momentum space is

$$i\,q^2 q^\mu\, D_{\mu\nu}(q) = \xi\, q_\nu .$$

E. Draw the Feynman diagrams for $D_{\mu\nu}(q)$ at order e^0 and at order e^2. Label each line with a momentum. Label the ends of the lines and each vertex with a Lorentz index.



F. The order-e^0 term in $D_{\mu\nu}(q)$ is the Feynman propagator:

$$D_{F\mu\nu}(q) = \frac{i\big[-g_{\mu\nu} + (1-\xi) q_\mu q_\nu / q^2\big]}{q^2 + i\epsilon}.$$

Verify the Ward identity at order e^0.

$$i q^2 q^\mu \, \frac{i\big[-g_{\mu\nu} + (1-\xi) q_\mu q_\nu / q^2\big]}{q^2 + i\epsilon} = \xi\, q_\nu$$

G. Verify the following identity:

$$\frac{1}{\not k + \not q - m + i\epsilon}(q_\alpha \gamma^\alpha)\frac{1}{\not k - m + i\epsilon} = \boxed{\frac{1}{\not k - m + i\epsilon} - \frac{1}{\not k + \not q - m + i\epsilon}}.$$

$$= \frac{1}{\not k + \not q - m + i\epsilon}\Big[(\not k + \not q - m) - (\not k - m)\Big]\frac{1}{\not k - m + i\epsilon} =$$

The order-e^2 term in $D_{\mu\nu}(q)$ is

$$\underbrace{(q^2 q^\mu D_{F\mu\alpha}(q)}_{\xi q_\alpha}\,(-1)\int \frac{d^4 k}{(2\pi)^4}\,\mathrm{Tr}\left(\frac{i}{\not k + \not q - m + i\epsilon}(-ie\gamma^\alpha)\frac{i}{\not k - m + i\epsilon}(-ie\gamma^\beta)\right) D_{F\beta\nu}(q).$$

The Ward identity requires its contraction with $iq^2 q^\mu$ to be zero.

H. Verify that the Ward identity is satisfied at order e^2 provided the loop momentum k can be shifted.

$$= -\xi e^2 \int \frac{d^4 k}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{1}{\not k + \not q - m + i\epsilon}(q_\alpha \gamma^\alpha)\frac{1}{\not k - m + i\epsilon}\gamma^\beta\right] D_{F\beta\nu}(q)$$

$$= -\xi e^2 \int \frac{d^4 k}{(2\pi)^4}\,\mathrm{Tr}\left[\left(\frac{1}{\not k - m + i\epsilon} - \frac{1}{\not k + \not q - m + i\epsilon}\right)\gamma^\beta\right] D_{F\beta\nu}(q)$$

$= 0$ if the loop momentum in 2nd term can be shifted: $k^\mu \to k^\mu - q^\mu$