Running Coupling Constant for QED

The complete propagator for a photon of momentum q has the form

$$D_{\mu\nu}(q) = \frac{i}{q^2[1-\Pi(q^2)]}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) - i\xi\frac{q_\mu q_\nu}{(q^2)^2}.$$

where $q^2\,\Pi(q^2)$ is the photon self-energy function and ξ is the covariant gauge-fixing parameter.

A. Verify the Ward identity $q^2 q^\mu D_{\mu\nu}(q) = -i\xi q_\nu$.

$$q^\mu\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) = -q_\nu + \frac{q^2 q_\nu}{q^2} = 0 \qquad q^\mu \frac{q_\mu q_\nu}{q^2} = \frac{q^2 q_\nu}{q^2} = q_\nu \implies q^2 q^\mu D_{\mu\nu}(q) = q^2\left(-i\frac{\xi}{q^2}\right)q_\nu$$
$$= -i\xi q_\nu$$

The self-energy tensor for a photon of momentum q has the form

$$\Pi^{\mu\nu}(q) = \Pi(q^2)\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right].$$

B. Verify the Ward identity $q_\mu \Pi^{\mu\nu}(q) = 0$.

$$q_\mu\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) = q^2 q^\nu - q^2 q^\nu = 0$$

The self-energy function can be expanded in powers of the bare coupling constant α_0:

$$\Pi(q^2) = \alpha_0\,\Pi_1(q^2) + \alpha_0^2\,\Pi_2(q^2) + \ldots.$$

C. Draw the one-loop diagram for $i\alpha_0\,\Pi_1(q^2)[q^2 g^{\mu\nu} - q^\mu q^\nu]$.

D. Draw the 3 two-loop diagrams for $i\alpha_0^2\,\Pi_2(q^2)[q^2 g^{\mu\nu} - q^\mu q^\nu]$.

A running coupling constant for QED can be defined by

$$\bar{\alpha}(Q) = \frac{\alpha_0}{1-\Pi(-Q^2)}.$$

E. Set $Q = m_e$ to obtain an equation that relates $\bar{\alpha}(m_e)$ and α_0.

$$\bar{\alpha}(m_e) = \frac{\alpha_0}{1-\Pi(-m_e^2)}$$

F. Solve the equation for α_0 to next-to-leading order in $\bar{\alpha}(m_e)$.

$$\alpha_0 = \bar{\alpha}(m_e)\left[1-\Pi(-m_e^2)\right] = \bar{\alpha}(m_e)\left[1-\alpha_0\Pi_1(-m_e^2) + O(\alpha_0^2)\right]$$
$$= \bar{\alpha}(m_e)\left[1-\bar{\alpha}(m_e)\Pi_1(-m_e^2) + O(\bar{\alpha}(m_e)^2)\right]$$

The 1-loop, 2-loop, and 3-loop self-energy functions for $|q^2| \gg m_e^2$ have the forms

$$\begin{aligned}
\alpha_0\,\Pi_1(q^2) &= \alpha_0\,\big[b_1 \log(-q^2/m_e^2) + \boxed{c_1}\big],\\
\alpha_0^2\,\Pi_2(q^2) &= \alpha_0^2\,\big[\boxed{b_2}\log(-q^2/m_e^2) + c_2\big],\\
\alpha_0^3\,\Pi_3(q^2) &= \alpha_0^3\,\big[\boxed{a_3}\log^2(-q^2/m_e^2) + b_3\log(-q^2/m_e^2) + c_3\big].
\end{aligned}$$

The coefficients c_1, c_2, b_3, and c_3 are ultraviolet divergent.

G. Draw a circle around the term that is a leading logarithm (as many powers of $\log(-q^2/m_e^2)$ as powers of α_0).

H. Draw a square around each term that is a next-to-leading logarithm (one fewer power of $\log(-q^2/m_e^2)$ than powers of α_0).

The geometric series of 1-loop self-energy diagrams includes all the leading logarithms of $-q^2/m_e^2$. The running coupling constant including these diagrams can be expressed as

$$1/\bar\alpha(Q) = \big[1 - \Pi_1(-Q^2)\big]/\alpha_0 = 1/\alpha_0 - \big[b_1\log(Q^2/m_e^2) + c_1\big].$$

I. Determine the 1-loop beta function by applying Qd/dQ to the first and last expressions.

$Q\frac{d}{dQ}\frac{1}{\bar\alpha(Q)} = -\frac{1}{\bar\alpha(Q)^2}\,Q\frac{d}{dQ}\bar\alpha(Q)$ $\quad Q\frac{d}{dQ}\left[\frac{1}{\alpha_0} - b_1\log\frac{Q^2}{m_e^2} - c_1\right] = -b_1\cdot 2 \Rightarrow Q\frac{d}{dQ}\bar\alpha = 2b_1\bar\alpha^2$

J. Set $Q = m_e$ to get an expression for $1/\alpha_0$. Use it to eliminate $1/\alpha_0$ and get an expression for $\alpha(Q)$ in the leading log approximation.

$1/\bar\alpha(m_e) = 1/\alpha_0 - [b_1\cdot 0 + c_1] \Rightarrow \bar\alpha(m_e) = \frac{1}{1/\alpha_0 - c_1} = \frac{\alpha_0}{1 - c_1\alpha_0}$

The running coupling constant including the geometric series of 1-loop and 2-loop self-energy diagrams can be expressed as

$$1/\bar\alpha(Q) = 1/\alpha_0 - \big[b_1\log(Q^2/m_e^2) + c_1\big] - \alpha_0\big[b_2\log(Q^2/m_e^2) + c_2\big].$$

K. Apply Qd/dQ to both sides of the equation.

$Q\frac{d}{dQ}\frac{1}{\bar\alpha(Q)} = -\frac{1}{\bar\alpha(Q)^2}\,Q\frac{d}{dQ}\bar\alpha(Q) = -b_1\cdot 2 - \alpha_0 b_2\cdot 2$

L. Assume that if the next-to-leading logarithms are summed to all orders, the beta function is a function of $\bar\alpha(Q)$ only. Use the result from part K to deduce the beta function to next-to-leading order in α.

$Q\frac{d}{dQ}\bar\alpha = 2b_1\bar\alpha^2 - 2b_2\bar\alpha^3$