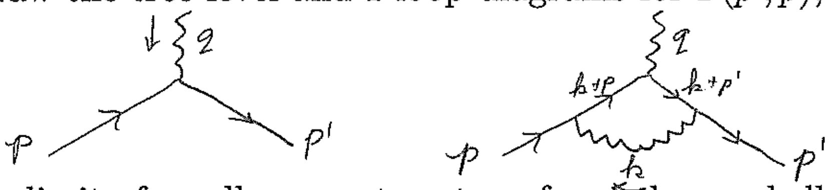
Renormalization of Magnetic Moment

The 1PI vertex for an electron to change momentum from p to p' by absorbing a virtual photon of momentum $q = p' - p$) is $ie_0\Gamma(p', p)$.

A. Draw the tree-level and 1-loop diagrams for $\Gamma(p', p)$, labelling the momenta.



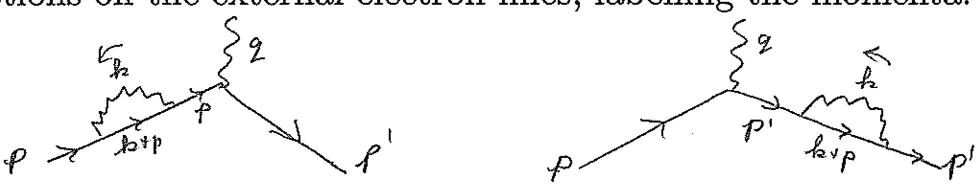
In the limit of small momentum transfer q, the on-shell IPI vertex sandwiched between electron spinors reduces to

$$ie_0\,\bar{u}(p')\Gamma(p', p)u(p) \longrightarrow ie_0\,\bar{u}(p')\left[h_0\,(p+p')^\mu + g_0\frac{i}{2m_e}\sigma^{\mu\nu}q_\nu\right]u(p).$$

where the coefficients are

$$\begin{aligned} h_0 &= 1 + \frac{e_0^2}{16\pi^2}\left(\frac{4}{d-4} - 5 + 4\log\frac{m_e}{\bar{\mu}}\right), \\ g_0 &= 2 + \frac{e_0^2}{16\pi^2}\left(\frac{8}{d-4} - 6 + 8\log\frac{m_e}{\bar{\mu}}\right). \end{aligned}$$

B. Draw the two 1-loop diagrams for corrections to the vertex from propagator corrections on the external electron lines, labelling the momenta.



The effect of each of these diagrams on the QED vertex is to multiply it by a factor of $\sqrt{Z_e}$, where Z_e is

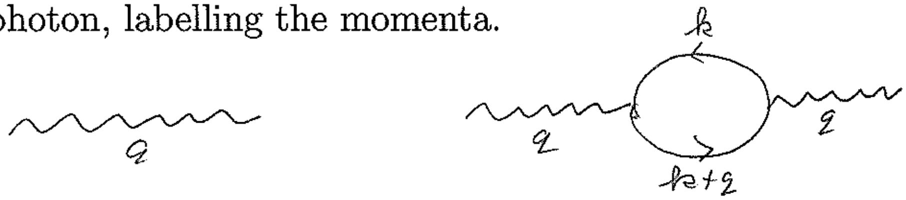
$$Z_e = 1 + \frac{e_0^2}{16\pi^2}\left(-\frac{4}{d-4} + 5 - 4\log\frac{m_e}{\bar{\mu}}\right).$$

C. Verify that $(\sqrt{Z_e})^2 h_0 = 1 + \mathcal{O}(e_0^4)$.

$$(\sqrt{Z_e})^2 h_0 = Z_e h_0 = 1 + \frac{e_0^2}{16\pi^2}\left(-\frac{4}{d-4} + 5 - 4\log\frac{m_e}{\bar{\mu}}\right) + \frac{e_0^2}{16\pi^2}\left(\frac{4}{d-4} - 5 + 4\log\frac{m_e}{\mu}\right) + \mathcal{O}(e_0^4)$$

$$= 1 + \mathcal{O}(e_0^4)$$

D. Draw the tree-level and 1-loop diagrams for the complete propagator of the virtual photon, labelling the momenta.



The sum of the tree-level and 1-loop diagrams for the complete photon propagator can be expressed as $\left[1+\Pi(q^2)\right](-ig_{\mu\nu}/q^2)$. The correction factor in the small-q^2 limit is

$$1+\Pi(0) = 1 + \frac{e_0^2}{6\pi^2}\left(\frac{1}{d-4} + \log\frac{m_e}{\bar{\mu}}\right).$$

The renormalization of the coupling constant at this order is $e_R^2 = e_0^2[1+\Pi(0)]$.
E. Express e_R as an expansion in powers of e_0 to order e_0^3.

$$e_R = e_0\sqrt{1+\Pi(0)} = e_0\left[1+\tfrac{1}{2}\Pi(0) + O(e_0^4)\right] = e_0\left[1+\frac{e_0^2}{12\pi^2}\left(\frac{1}{d-4}+\log\frac{m_e}{\bar{\mu}}\right)\right]$$

The effect of the photon propagator correction on the QED vertex in the small-q limit is to multiply it by a factor of $\sqrt{Z_\gamma}$, where $Z_\gamma = 1+\Pi(0)$.
F. Verify that $\sqrt{Z_\gamma}\, e_0 = e_R[1+\mathcal{O}(e_R^4)]$.

$$\sqrt{Z_\gamma}\, e_0 = \sqrt{1+\Pi(0)}\, e_0 = e_R + O(e_R^4)$$

In the limit of small momentum transfer q, the complete on-shell QED vertex sandwiched between electron spinors reduces to

$$ie_0\,(\sqrt{Z_e})^2\sqrt{Z_\gamma}\,\bar{u}(p')\Gamma(p',p)u(p) \longrightarrow ie_R\,\bar{u}(p')\left[h\,(p+p')^\mu + g\frac{i}{2m_e}\sigma^{\mu\nu}q_\nu\right]u(p),$$

where q is the electric charge and g is the gyromagnetic ratio.
G. Verify that q and g satisfy

$$\begin{aligned}(\sqrt{Z_e})^2\sqrt{Z_\gamma}\,h_0\,e_0 &= h\,e_R, \quad ✓\\ (\sqrt{Z_e})^2\sqrt{Z_\gamma}\,g_0\,e_0 &= g\,e_R. \quad ✓\end{aligned}$$

H. Use parts C and F to verify that $h = 1+\mathcal{O}(e_R^4)$.

$$he_R = (\sqrt{Z_e})^2 h_0\cdot\sqrt{Z_\gamma}\,e_0 = \left[1+O(e_0^4)\right]e_R\left[1+O(e_R^4)\right] = e_R\left[1+O(e_R^4)\right]$$

$$\Longrightarrow h = 1+O(e_R^4)$$

I. Calculate $(\sqrt{Z_e})^2 g_0$ to order e_0^2.

$$(\sqrt{Z_e})^2 g_0 = \left[1+\frac{e_0^2}{16\pi^2}\left(-\frac{4}{d-4}+5-4\log\frac{m_e}{\bar{\mu}}\right)\right]\left[2+\frac{e_0^2}{16\pi^2}\left(\frac{8}{d-4}-6+8\log\frac{m_e}{\bar{\mu}}\right)\right]$$

$$= 2+\frac{e_0^2}{16\pi^2}(10-6)+O(e_0^4) = 2+\frac{e_0^2}{4\pi^2}+O(e_0^4)$$

J. Calculate g to order e_R^2.

$$ge_R = (\sqrt{Z_e})^2 g_0\cdot\sqrt{Z_\gamma}\,e_0 = \left(2+\frac{e_0^2}{4\pi^2}\right)e_R\left[1+O(e_0^4)\right] \Longrightarrow g = 2+\frac{e_R^2}{4\pi^2}+O(e_R^4)$$