Wavefunction Renormalization for a Scalar Field

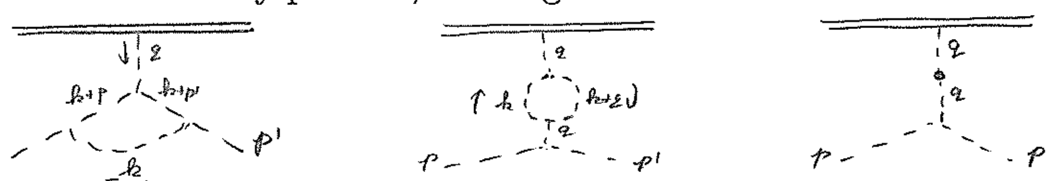
The Lagrangian for a real scalar field with a cubic interaction is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{6}g\phi^3 - \tfrac{1}{2}\delta m^2\phi^2,$$

where δm^2 is order g^2. The Feynman rules are $i/(p^2 - m^2 + i\epsilon)$ for the scalar propagator, $-ig$ for the 3-scalar vertex, and $-i\delta m^2$ for the 2-scalar vertex. The scalar also couples to a heavy particle with a vertex $-iG$.

A. Draw the tree-level Feynman diagram of order g for the scattering of the scalar from the heavy particle, with momentum $q = p' - p$ transferred to the scalar. Write down the matrix element $i\mathcal{M}$.

B. Draw the three Feynman diagrams of order g^3 for the scattering of the scalar from the heavy particle, labeling the momenta of the scalar lines.



C. The contribution to $i\mathcal{M}$ from the vertex-correction diagram has the form $(-iG)(-ig)^3[i/(q^2 - m^2)]F(q^2)$. Express the form factor $F(q^2)$ as an integral over a loop momentum.

$$F(q^2) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k+p)^2 - m^2 + i\epsilon} \frac{i}{(k+p')^2 - m^2 + i\epsilon}$$

The contribution to $i\mathcal{M}$ from the two propagator-correction diagrams is

$$(-iG)(-ig)\frac{i}{q^2 - m^2}\Big[- i\Pi(q^2) - i\delta m^2\Big]\frac{i}{q^2 - m^2},$$

where the self-energy function is

$$\Pi(p^2) = -\frac{g^2}{32\pi^2}\left(\log\frac{\Lambda^2}{m^2} - 1 + \int_0^1 dx\, \log\frac{m^2 - x(1-x)p^2 - i\epsilon}{m^2}\right).$$

D. How must δm^2 depend on the ultraviolet cutoff Λ so that $\mathcal{M}$ does not depend on Λ? $\delta m^2 = \frac{g^2}{32\pi^2}\left(\log\frac{\Lambda^2}{m^2} + \text{constant}\right)$

E. What must δm^2 be to avoid an unphysical double pole $1/(q^2 - m^2)^2$?

$$\delta m^2 = -\Pi(m^2) = \frac{g^2}{32\pi^2}\left[\log\frac{\Lambda^2}{m^2} - 1 + \int_0^1 dx\, \log(1 - x + x^2)\right]$$

The self-energy function and its derivative at $p^2 = m^2$ are

$$\Pi(m^2) = -\frac{g^2}{32\pi^2}\left(\log\frac{\Lambda^2}{m^2} + \frac{\pi}{\sqrt{3}} - 3\right), \qquad \Pi'(m^2) = \frac{g^2}{32\pi^2 m^2}\left(\frac{2\pi}{3\sqrt{3}} - 1\right).$$

The complete propagator obtained by summing the geometric series of order-g^2 self-energy diagrams to all orders is

$$D(p^2) = \frac{i}{p^2 - m^2 - \Pi(p^2) - \delta m^2 + i\epsilon}.$$

The pole in $D(p^2)$ is at $p^2 = m^2_{\text{phys}}$, where m_{phys} is the physical mass.

F. Write down the equation that must be satisfied by m^2_{phys}.

$$m_{phys}^2 - m^2 - \Pi(m_{phys}^2) - \delta m^2 = 0$$

G Suppose $\delta m^2 = (g^2/32\pi^2)\log(\Lambda^2/m^2)$? Solve the equation for m^2_{phys} to order g^2. (In terms with an explicit factor of g^2, you can replace m^2_{phys} by m^2.)

$$m_{phys}^2 = m^2 + \Pi(m^2) + \delta m^2 = m^2 - \frac{g^2}{32\pi^2}\left(\log\frac{\Lambda^2}{m^2} + \frac{\pi}{\sqrt{3}} - 3\right) + \frac{g^2}{32\pi^2}\log\frac{\Lambda^2}{m^2}$$

$$= m^2 - \frac{g^2}{32\pi^2}\left(\frac{\pi}{\sqrt{3}} - 3\right)$$

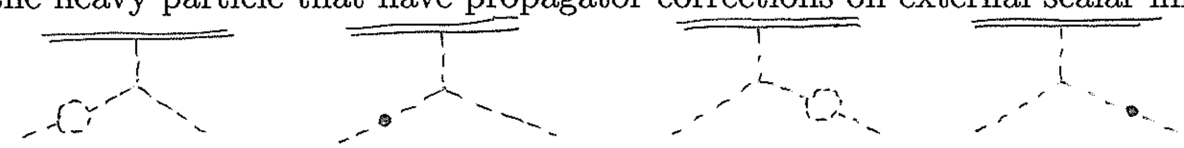
H. Suppose $\delta m^2 = -\Pi(m^2)$. What is the value of m_{phys}?

$$m_{phys}^2 = m^2$$

I. Having chosen δm^2 so m is the physical mass, determine the residue iZ_s of the pole in $D(p^2)$ at $p^2 = m^2$, and expand Z_s to order g^2.

$$Z_s = \frac{1}{1 - \Pi'(m^2)} = \frac{1}{1 - \frac{g^2}{32\pi^2 m^2}\left(\frac{2\pi}{3\sqrt{3}} - 1\right)} = 1 + \frac{g^2}{32\pi^2 m^2}\left(\frac{2\pi}{3\sqrt{3}} - 1\right)$$

J. Draw the ~~two~~ four Feynman diagrams of order g^3 for scattering of the scalar from the heavy particle that have propagator corrections on external scalar lines.



The complete matrix element for scattering at next-to-leading order in g is

$$\mathcal{M} = \left(\sqrt{Z_s}\right)^2 \frac{Gg}{q^2 - m^2}\left[1 + g^2 F(q^2) + \frac{\Pi(q^2) - \Pi(m^2)}{q^2 - m^2} + \mathcal{O}(g^4)\right].$$

K. Expand this out to obtain the complete order g^3 contribution to $\mathcal{M}$.

$$\mathcal{M} = \frac{Gg}{q^2 - m^2} + \frac{Gg^3}{q^2 - m^2}\left[F(q^2) + \frac{\Pi(q^2) - \Pi(m^2)}{g^2(q^2 - m^2)} + \frac{1}{32\pi^2 m^2}\left(\frac{2\pi}{3\sqrt{3}} - 1\right)\right]$$