Wavefunction Renormalization for a Scalar Field

The Lagrangian for a real scalar field with a cubic interaction is

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{6} g \phi^3 - \frac{1}{2} \delta m^2 \phi^2,$$

where $\delta m^2$ is order $g^2$. The Feynman rules are $i/(p^2 - m^2 + i\epsilon)$ for the scalar propagator, $-ig$ for the 3-scalar vertex, and $-i\delta m^2$ for the 2-scalar vertex. The scalar also couples to a heavy particle with a vertex $-iG$.

A. Draw the tree-level Feynman diagram of order $g$ for the scattering of the scalar from the heavy particle, with momentum $q = p' - p$ transferred to the scalar. Write down the matrix element $i\mathcal{M}$. 

B. Draw the three Feynman diagrams of order $g^3$ for the scattering of the scalar from the heavy particle, labeling the momenta of the scalar lines.

C. The contribution to $i\mathcal{M}$ from the vertex-correction diagram has the form $(-iG)(-ig)^3[i/(q^2 - m^2)]F(q^2)$. Express the form factor $F(q^2)$ as an integral over a loop momentum.

$$F(q^2) = \int_0^{2\pi} \frac{d^2 \ell}{(2\pi)^2} \frac{\epsilon}{(k', p')^2 - m^2 + i\epsilon} \frac{\epsilon}{(k, p)^2 - m^2 + i\epsilon} \frac{\epsilon}{(k, p)^2 - m^2 + i\epsilon}$$

The contribution to $i\mathcal{M}$ from the two propagator-correction diagrams is

$$(-iG)(-ig) \frac{i}{q^2 - m^2} \left[ -i\Pi(q^2) - i\delta m^2 \right] \frac{i}{q^2 - m^2},$$

where the self-energy function is

$$\Pi(p^2) = -\frac{g^2}{32\pi^2} \left( \log \frac{\Lambda^2}{m^2} - 1 + \int_0^1 dx \log \frac{m^2 - x(1-x)p^2 - i\epsilon}{m^2} \right).$$

D. How must $\delta m^2$ depend on the ultraviolet cutoff $\Lambda$ so that $\mathcal{M}$ does not depend on $\Lambda$?

$$S_{\delta m^2} = \frac{g^2}{32\pi^2} \left( \log \frac{\Lambda^2}{m^2} + \text{constant} \right)$$

E. What must $\delta m^2$ be to avoid an unphysical double pole $1/(q^2 - m^2)^2$?

$$S_{\delta m^2} = -\Pi(m^2) = \frac{g^2}{32\pi^2} \left[ \log \frac{\Lambda^2}{m^2} - 1 + \int_0^1 dx \log \left( \frac{\Lambda^2}{m^2} - 1 \right) \right]$$
The self-energy function and its derivative at \( p^2 = m^2 \) are

\[
\Pi(m^2) = -\frac{g^2}{32\pi^2} \left( \log \frac{\Lambda^2}{m^2} + \frac{\pi}{\sqrt{3}} - 3 \right), \quad \Pi'(m^2) = \frac{g^2}{32\pi^2 m^2} \left( \frac{2\pi}{3\sqrt{3}} - 1 \right).
\]

The complete propagator obtained by summing the geometric series of order-\( g^2 \) self-energy diagrams to all orders is

\[
D(p^2) = \frac{i}{p^2 - m^2 - \Pi(p^2) - \delta m^2 + i\epsilon}.
\]

The pole in \( D(p^2) \) is at \( p^2 = m_{\text{phys}}^2 \), where \( m_{\text{phys}} \) is the physical mass.

F. Write down the equation that must be satisfied by \( m_{\text{phys}}^2 \).

\[
r\Pi_{\text{phys}} - m^2 - \Pi(m_{\text{phys}}^2) - \delta m^2 = 0
\]

G. Suppose \( \delta m^2 = (g^2/32\pi^2) \log(\Lambda^2/m^2) \)? Solve the equation for \( m_{\text{phys}}^2 \) to order \( g^2 \). (In terms with an explicit factor of \( g^2 \), you can replace \( m_{\text{phys}}^2 \) by \( m^2 \).)

\[
m_{\text{phys}}^2 = m^2 + \Pi(m^2) + \delta m^2 = m^2 - \frac{g^2}{32\pi^2} \left( \log \frac{\Lambda^2}{m^2} + \frac{\pi}{\sqrt{3}} - 3 \right) + \frac{g^2}{32\pi^2} \log \frac{\Lambda^2}{m^2} = m^2 - \frac{g^2}{32\pi^2} \left( \frac{\pi}{\sqrt{3}} - 3 \right)
\]

H. Suppose \( \delta m^2 = -\Pi(m^2) \). What is the value of \( m_{\text{phys}} \)?

\[
m_{\text{phys}}^2 = m^2
\]

I. Having chosen \( \delta m^2 \) so \( m \) is the physical mass, determine the residue \( iZ_s \) of the pole in \( D(p^2) \) at \( p^2 = m^2 \), and expand \( Z_s \) to order \( g^2 \).

\[
Z_s = \frac{i}{l - \Pi(m^2)} = \frac{i}{1 - \frac{g^2}{32\pi^2 m^2} \left( \frac{2\pi}{3\sqrt{3}} - 1 \right)} = 1 + \frac{g^2}{32\pi^2 m^2} \left( \frac{2\pi}{3\sqrt{3}} - 1 \right)
\]

J. Draw the two Feynman diagrams of order \( g^3 \) for scattering of the scalar from the heavy particle that have propagator corrections on external scalar lines.

\[
\begin{align*}
\text{Diagram 1} & \quad \text{Diagram 2} \quad \text{Diagram 3}
\end{align*}
\]

The complete matrix element for scattering at next-to-leading order in \( g \) is

\[
M = \left( \sqrt{Z_s} \right)^2 \frac{G}{q^2 - m^2} \left[ 1 + g^2 F(q^2) + \frac{\Pi(q^2) - \Pi(m^2)}{q^2 - m^2} + \mathcal{O}(g^4) \right].
\]

K. Expand this out to obtain the complete order \( g^3 \) contribution to \( M \).

\[
\mathcal{M} = \frac{G}{q^2 - m^2} + \frac{G g^3}{2 \sqrt{2} m^3} \left[ F(q^2) + \frac{\Pi(q^2) - \Pi(m^2)}{q^2 - m^2} + \frac{1}{32\pi^2 m^2} \left( \frac{2\pi}{3\sqrt{3}} - 1 \right) \right]
\]