Ward Identity for Gauge-fixing Renormalization

The photon terms in the Lagrangian for QED with a covariant gauge-fixing term with bare gauge parameter ξ_0 are

$$\mathcal{L}_{\text{photon}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi_0}(\partial^\mu A_\mu)^2.$$

The Ward identity for the complete coordinate-space photon propagator is

$$\partial_x^\mu \partial_x^2 \langle A_\mu(x) A_\nu(y) \rangle = i\xi_0 \frac{\partial}{\partial x^\nu}\delta^4(x-y).$$

A. Fourier transform in both coordinates by applying $\int d^4x\, e^{iq.x} \int d^4y\, e^{iq'.y}$ to both sides of the equation. Eliminate derivatives ∂_x^μ in favor of factors of q^μ.

$$\int d^4x\, e^{iq\cdot x} \int d^4y\, e^{iq'\cdot x}\, \partial_x^\mu \partial_x^2 \langle A_\mu(x) A_\nu(y)\rangle = i\xi_0 \int d^4x\, e^{iq\cdot x}\int d^4y\, e^{iq'\cdot y} \frac{\partial}{\partial x_\nu}\delta^4(x-y)$$

$$(-iq^\mu)(-iq)^2 \int d^4x\, e^{iq\cdot x}\int d^4y\, e^{iq'\cdot x}\langle A_\mu(x) A_\nu(y)\rangle = i\xi_0(-iq_\nu)\int d^4x\, e^{iq\cdot x}\int d^4y\, e^{iq'\cdot y}\delta^4(x-y)$$

$$i q^\mu \langle \tilde{A}_\mu(q)\tilde{A}_\nu(q')\rangle = \xi_0 q_\nu \int d^4x\, e^{i(q+q')\cdot x} = \xi_0 q_\nu (2\pi)^4 \delta^4(q+q')$$

The complete momentum-space photon propagator can be expressed as

$$D_{\mu\nu}(q) = \int d^4x\, e^{iq.(x-y)}\, \langle A_\mu(x) A_\nu(y)\rangle.$$

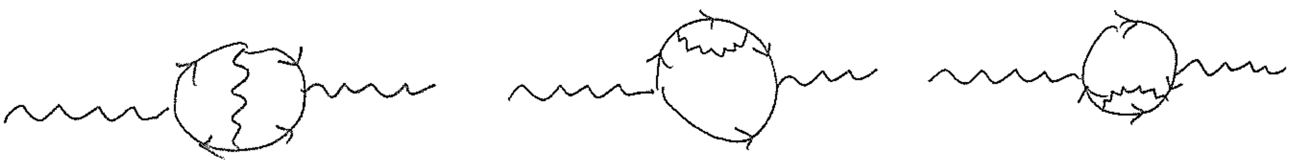
B. Factor out a delta function $(2\pi)^4\delta^4(q+q')$ from the equation in part A to get the Ward identity for $D_{\mu\nu}(q)$.

$$\langle \tilde{A}_\mu(q)\tilde{A}_\nu(q')\rangle = \int d^4x\, e^{iq\cdot x}\int d^4y\, e^{iq'\cdot x}\langle A_\mu(x) A_\nu(y)\rangle$$
$$= \int d^4y\, e^{i(q+q')\cdot y}\int d^4x\, e^{iq'\cdot(x-y)}\langle A_\mu(x) A_\nu(y)\rangle = (2\pi)^4\delta^4(q+q')\, D_{\mu\nu}(q)$$

$$iq^\mu (2\pi)^4\delta^4(q+q')\, D_{\mu\nu}(q) = \xi_0 q\, (2\pi)^4\delta^4(q+q') \qquad iq^\mu D_{\mu\nu}(q) = \xi_0 q_\nu$$

C. Draw the one-loop diagram for the photon self-energy $\Pi^{\mu\nu}(q)$ in bare perturbation theory.

D. Draw the 3 two-loop diagrams for $\Pi^{\mu\nu}(q)$ in bare perturbation theory.



The Lagrangian for QED in renormalized perturbation theory can be obtained by renormalizing the parameters (including $\xi_0 = Z_\xi \xi$) and rescaling the fields (including $A_\mu(x) \to \sqrt{Z_3} A_\mu(x)$). The photon terms in the Lagrangian become

$$\mathcal{L}_{\rm photon} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} - \frac{1}{2 Z_\xi \xi} Z_3 (\partial^\mu A_\mu)^2 .$$

E. Separate this into a conventional photon kinetic term with renormalized gauge-fixing parameter ξ and a counterterm Lagrangian.

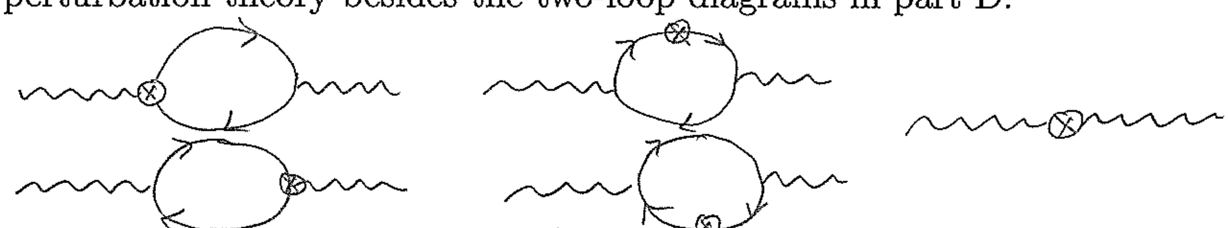
$$\mathcal{L}_{\rm photon} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2$$
$$- \frac{1}{4}(Z_3 - 1) F_{\mu\nu} F^{\mu\nu} - \left(\frac{Z_3}{Z_\xi} - 1\right) \frac{1}{2\xi} (\partial^\mu A_\mu)^2$$

The photon self-energy counterterm for a photon of momentum q is

$$i(Z_3 - 1)(q^2 g^{\mu\nu} - q^\mu q^\nu) + i(Z_3/Z_\xi - 1)\xi q^\mu q^\nu .$$

F. Draw the additional diagram for $\Pi^{\mu\nu}(q)$ at order e^2 in renormalized perturbation theory besides the one-loop diagram in part C.

G. Draw the 5 additional diagrams for $\Pi^{\mu\nu}(q)$ at order e^4 in renormalized perturbation theory besides the two-loop diagrams in part D.



H. Obtain the Ward identity for $\langle A_\mu(x) A_\nu(y) \rangle$ in renormalized perturbation theory by renormalizing the gauge parameter ($\xi_0 = Z_\xi \xi$) and rescaling the field ($A_\mu(x) \to \sqrt{Z_3} A_\mu(x)$).

$$Z_3 \, \partial_x^\mu \partial_x^2 \langle A_\mu(x) A_\nu(y) \rangle = i Z_\xi \xi \frac{\partial}{\partial x^\nu} \delta^4(x-y)$$

I. Give the Ward identity for $D_{\mu\nu}(q)$ in renormalized perturbation theory.

$$Z_3 \, q^\mu D_{\mu\nu}(q) = Z_\xi \xi q_\nu$$

J. Use the finiteness of $D_{\mu\nu}(q)$ in renormalized perturbation theory to deduce that Z_ξ / Z_3 must be a power series in α with finite coefficients.

multiply by $1/Z_3$: $\quad q^\mu D_{\mu\nu}(q) = \dfrac{Z_\xi}{Z_3} \xi q_\nu$

left side is finite $\Longrightarrow$ right side is finite $\Longrightarrow$ $\dfrac{Z_\xi}{Z_3}$ is finite