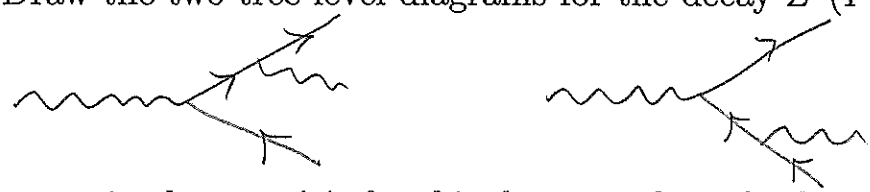
Soft Photons in Z^0 Decay (continued)

The leading-order matrix element $\mathcal{M}_0$ for the decay $Z^0(P) \to \mu^-(p_1)\mu^+(p_2)$ is

$$\mathcal{M}_{\rm LO} = e\,\varepsilon_\mu(P)\,\bar{u}(p_1)(g_V - g_A\gamma_5)\gamma^\mu v(p_2).$$

A. Draw the two tree-level diagrams for the decay $Z^0(P) \to \mu^-(p_1)\mu^+(p_2)\gamma(q)$.



The matrix element $\mathcal{M}_1$ for this decay in the soft-photon limit is

$$\mathcal{M}_1 = \mathcal{M}_{\rm LO}\, e\left(\frac{p_1.\varepsilon(q)}{p_1.q} - \frac{p_2.\varepsilon(q)}{p_2.q}\right)$$

B. Verify the Ward identity that states that $\mathcal{M}_1 = 0$ if $\varepsilon^\mu(q)$ is replaced by q^μ.

$$\mathcal{M}_1\Big|_{\varepsilon^\mu \to q^\mu} = \mathcal{M}_{LO}\, e\left(\frac{p_1\cdot q}{p_1\cdot q} - \frac{p_2\cdot q}{p_2\cdot q}\right) = 0$$

The polarization sum for the photon is

$$\sum_{\rm photons} \varepsilon_\mu(q)\varepsilon_\nu(q)^* = -g_{\mu\nu} + \big(\text{terms with } q_\mu \text{ or } q_\nu\big).$$

C. Use this to simplify $\sum_{\rm photons}|\mathcal{M}_1|^2$.

$$\sum_{\substack{\text{photon}\\\text{spins}}} |\mathcal{M}_1|^2 = |\mathcal{M}_{L0}|^2 e^2 \left(\frac{p_1^\alpha}{p_1\cdot q} - \frac{p_2^\alpha}{p_2\cdot q}\right)\left(\frac{p_1^\beta}{p_1\cdot q} - \frac{p_2^\beta}{p_2\cdot q}\right)\overbrace{\sum_{\rm spins} \varepsilon_\alpha(q)\,\varepsilon^*_\beta(q)}^{-g_{\alpha\beta}}$$

$$= |\mathcal{M}_{L0}|^2\, e^2\left[-\frac{m^2}{(p_1\cdot q)^2} - \frac{m^2}{(p_2\cdot q)^2} + 2\frac{p_1\cdot p_2}{p_1\cdot q\; p_2\cdot q}\right]$$

In the soft-photon limit, the energy-momentum conserving delta function becomes independent of q:

$$(2\pi)^4\delta^4(P - p_1 - p_2 - q) \longrightarrow (2\pi)^4\delta^4(P - p_1 - p_2).$$

D. Express the differential decay rate $d\Gamma[\mu^-\mu^+\gamma]$ in the soft-photon limit as the differential decay rate $d\Gamma[\mu^-\mu^+]$ multiplied by a soft-photon factor.

$$d\Gamma[\mu^-\mu^+\gamma] = \frac{1}{2M_Z}|\mathcal{M}_{L0}|^2 e^2\left[\frac{2p_1\cdot p_2}{p_1\cdot q\, p_2\cdot q} - \frac{m^2}{(p_1\cdot q)^2} - \frac{m^2}{(p_2\cdot q)^2}\right](2\pi)^4\delta^4(P-p_1-p_2)\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3q}{(2\pi)^3}$$

$$= d\Gamma[\mu^+\mu^-]\, e^2\left[\frac{2p_1 p_2}{p_1\cdot q\, p_2\cdot q} - \frac{m^2}{(p_1\cdot q)^2} - \frac{m^2}{(p_2\cdot q)^2}\right]\frac{d^3q}{(2\pi)^3 2q}$$

If dimension regularization is used to regularize the IR divergence, the soft-photon integration measure is $\int d^{D-1}q = \int_0^{E_{\rm res}} q^{D-2}dq \int d\Omega_{D-1}$.

E. Verify that the integral over q gives a pole in $D - 4$.

$$\int_0^{E_{res}} \frac{q^{D-2}\,dq}{q}\,\frac{1}{q^2} = \int_0^{E_{res}} q^{D-5}\,dq = \frac{1}{D-4}\,q^{D-4}\Big|_0^{E_{res}} = \frac{E_{res}^{D-4}}{D-4}$$

The differential decay rate for $\mu^-\mu^+\gamma$ in the soft-photon limit is

$$d\Gamma[\mu^-\mu^+ + \text{ soft } \gamma] = d\Gamma_{\rm LO}[\mu^-\mu^+] \times \frac{e^2}{8\pi^2}\frac{E_{\rm res}^{D-4}}{(D-4)_{ir}}$$
$$\times \left(\frac{M_Z^2}{1-(\hat{p}\cdot\hat{q})^2} - \frac{m_\mu^2}{(1-\hat{p}\cdot\hat{q})^2} - \frac{m_\mu^2}{(1+\hat{p}\cdot\hat{q})^2}\right) d\Omega_{D-1}$$

where $\hat{p}$ is the unit vector in the direction of the μ^- in the CM frame.

F. Draw the two diagrams of order e^2 in renormalized perturbation theory for the muon self-energy $-i\Sigma(\not{p})$.

The derivative of $\Sigma(\not{p})$ with respect to $\not{p}$ at $\not{p} = 0$ with $m_\mu = 0$ is

$$\Sigma'(\not{p} = 0) = -\frac{e^2}{8\pi^2}\cancel{\bar{\mu}^{4-D}}\left[\frac{1}{(D-4)_{uv}} - \frac{1}{(D-4)_{ir}}\right] + \delta_2.$$

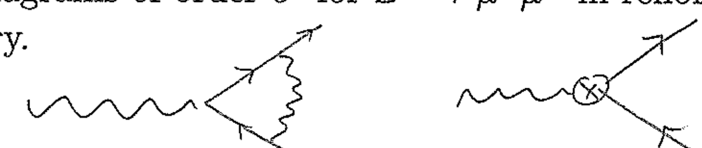
G. What is the counterterm δ_2 in the minimal subtraction (MS) renormalization scheme.

$$\delta_2 = \frac{e^2}{8\pi^2}\frac{1}{(D-4)_{uv}}$$

H. Determine the residue $Z_\mu = 1/[1-\Sigma'(\not{p}=0)]$ in the muon propagator to order e^2.

$$Z_\mu \simeq 1 + \Sigma'(\not{p}=0) = 1 + \frac{e^2}{8\pi^2}\frac{1}{(D-4)_{ir}}$$

I. Draw the two diagrams of order e^2 for $Z^0 \to \mu^-\mu^+$ in renormalized perturbation theory.



The matrix element $\mathcal{M}_2$ for $Z^0 \to \mu^-\mu^+$ at order e^2 with $m_\mu = 0$ has the form

$$\mathcal{M}_2 = \mathcal{M}_0\left(\frac{e^2}{8\pi^2}\left(\frac{\bar{\mu}}{M_Z}\right)^{4-D}\left[\frac{1}{(D-4)_{uv}} + \frac{a}{(D-4)_{ir}^2} + \frac{b}{(D-4)_{ir}} + c\right] + \delta_1\right)$$

J. What is the counterterm δ_1 in the MS renormalization scheme.

$$\delta_1 = \frac{e^2}{8\pi^2}\frac{1}{(D-4)_{uv}}$$

K. The differential decay rate for $Z^0 \to \mu^-\mu^+$ is proportional to $Z_\mu|\mathcal{M}_0 + \mathcal{M}_2|^2$. Express the order e^2 term as the leading-order rate $d\Gamma_{\rm LO}$ multiplied by a factor.

$$d\Gamma_1 = d\Gamma_{LO}\left|1 + \frac{e^2}{8\pi^2}\left[-\log\frac{\bar{\mu}}{M_Z} + \frac{a}{(D-4)_{ir}^2} + \frac{b+\frac{1}{2}}{(D-4)_{ir}} + c\right]\right|^2$$