Path Integral for EM Field

Lagrangian for electromagnetism

$$\mathcal{L}_{EM} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

classical electromagnetic current: $J_\mu(x)$

action: $$S[A] = \int d^4x \left[\mathcal{L}_{EM} + J^\mu A_\mu \right]$$

$$= \int d^4x \left[\tfrac{1}{2} A_\mu \mathcal{O}^{\mu\nu} A_\nu + J^\mu A_\mu \right]$$

where $\mathcal{O}^{\mu\nu} = -\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu$

path integral

$$Z[J] = \int \mathcal{D}A \, \exp\left(i \int d^4x \left[\tfrac{1}{2} A_\mu \mathcal{O}^{\mu\nu} A_\nu + J^\mu A_\mu \right] \right)$$

complete the square

$$= \int \mathcal{D}A \, \exp\left(i \int d^4x \left[\tfrac{1}{2} \left(A_\mu + J^\alpha \overleftarrow{\mathcal{O}}^{-1}_{\alpha\mu} \right) \mathcal{O}^{\mu\nu} \left(A_\nu + \mathcal{O}^{-1}_{\nu\beta} J^\beta \right) \right.\right.$$

$$\left.\left. - \tfrac{1}{2} J^\alpha \mathcal{O}^{-1}_{\alpha\beta} J^\beta \right] \right)$$

shift the field: $A_\mu \to A_\mu - \mathcal{O}^{-1}_{\nu\beta} J^\beta$

$$Z[J] = \int \mathcal{D}A \, \exp\left(i \int d^4x \left[\tfrac{1}{2} A_\mu \mathcal{O}^{\mu\nu} A_\nu - \tfrac{1}{2} J^\alpha \mathcal{O}^{-1}_{\alpha\beta} J^\beta \right] \right)$$

$$= Z[0] \exp\left(-i \int d^4x \, J^\alpha \mathcal{O}^{-1}_{\alpha\beta} J^\beta \right)$$

<u>problem</u>: $\mathcal{O}^{\mu\nu} = -g^{\mu\nu}\partial^2 + \partial^\mu \partial^\nu$ is <u>not</u> invertible!

for any scalar function $\theta(x)$,

$\partial_\nu \theta(x)$ is a zero eigenvector

$$\mathcal{O}^{\mu\nu} \partial_\nu \theta = (-g^{\mu\nu}\partial^2 + \partial^\mu \partial^\nu)\partial_\nu \theta$$

$$= -\partial^2 \partial^\mu \theta + \partial^\mu \partial^2 \theta = 0$$

gauge invariance:

$\mathcal{L}_{EM}$ is invariant under $A_\mu \longrightarrow A_\mu + \partial_\mu \theta$

solution by <u>gauge fixing</u>

add gauge-fixing term to Lagrangian

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

action in the presence of classical current

$$S[A] = \int d^4x \quad \mathcal{L}_{EM} + \mathcal{L}_{gf} + J^\mu A_\mu$$

$$= \int d^4x \left[\frac{1}{2} A_\mu \mathcal{O}^{\mu\nu} A_\nu + J^\mu A_\mu \right]$$

where $\mathcal{O}^{\mu\nu} = -g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu - \frac{1}{\xi}\partial^\mu\partial^\nu$

express operator as integral transform

$$\mathcal{O}^{\mu\nu} A_\nu(x) = \left(-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu - \frac{1}{\xi}\partial^\mu\partial^\nu\right) \int d^4x' \; \delta^4(x-x') \; A_\nu(x')$$

$$= \left(-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu - \frac{1}{\xi}\partial^\mu\partial^\nu\right) \int d^4x' \left[\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} \right] A_\nu(x')$$

$$= \int d^4x' \left[\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} \left(k^2 g^{\mu\nu} - k^\mu k^\nu + \frac{1}{\xi} k^\mu k^\nu \right) \right] A_\nu(x')$$

$$= \int d^4x' \, \mathcal{O}^{\mu\nu}(x,x') A_\nu(x')$$

$$\mathcal{O}^{\mu\nu}(x,x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \left[k^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{k^2}{\xi}\cdot\frac{k^\mu k^\nu}{k^2} \right]$$

integral transform for inverse operator
(with Feynman prescription)

$$\mathcal{O}^{-1}_{\alpha\beta}(x,x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} \left[\frac{1}{k^2+i\epsilon}\left(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) + \frac{\xi}{k^2+i\epsilon} \frac{k_\alpha k_\beta}{k^2} \right]$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} \frac{1}{k^2+i\epsilon}\left(g_{\alpha\beta} - (1-\xi)\frac{k_\alpha k_\beta}{k^2}\right)$$

photon propagator

$$\langle \Omega | T A_\mu(x) A_\nu(y) | \Omega \rangle$$

$$= \frac{1}{Z[J]} \left(-i \frac{\delta}{\delta J^\mu(x)}\right)\left(-i \frac{\delta}{\delta J^\nu(y)}\right) Z[J] \Big|_{J=0}$$

$$Z[J] = Z[0] \exp\left(-i \int d^4x \int d^4x' \, \frac{1}{2} J^\alpha(x) \, \mathcal{O}^{-1}_{\alpha\beta}(x,x') \, J^\beta(x') \right)$$

$$\frac{\delta}{\delta J^\mu(x)} \frac{\delta}{\delta J^\nu(y)} Z[J] \Big|_{J=0} = Z[0] \left(-i \, \mathcal{O}^{-1}_{\mu\nu}(x,y) \right)$$

$$\langle \Omega | T A_\mu(x) A_\nu(y) | \Omega \rangle = i \, \mathcal{O}^{-1}_{\mu\nu}(x,y)$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

Feynman propagator!

EM Lagrangian: $\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

gauge-fixing term: $\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$

gauge-invariant operators:

products of $F_{\mu\nu}(x)$, $\phi^*\phi(x)$

(but not $A_\mu(x)$, $\phi(y)$ or $\phi^*(z)$)

Show that for gauge-invariant operator $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A_\mu \, e^{i\int d^4x \, \mathcal{L}_{EM}} \, \mathcal{O}}{\int \mathcal{D}A_\mu \, e^{i\int d^4x \, \mathcal{L}_{EM}}}$$

$$= \frac{\int \mathcal{D}A_\mu \, e^{i\int d^4x (\mathcal{L}_{EM} + \mathcal{L}_{gf})} \, \mathcal{O}}{\int \mathcal{D}A_\mu \, e^{i\int d^4x (\mathcal{L}_{EM} + \mathcal{L}_{gf})}}$$

proof

for each photon field $A_\mu(x)$

define a field $\alpha_A(x)$ by $\Box \alpha_A = \partial_\mu A^\mu$

multiply numerator and denominator of each integral
by the same constant in the form of
a Gaussian integral over a scalar function $\pi(x)$

$$\int \mathcal{D}\pi \exp\left(-i\int d^4x \frac{1}{2\xi}(\Box\pi)^2\right) \int \mathcal{D}A \exp\left(i\int d^4x \mathcal{L}_{EM}\right)$$

$$= \int \mathcal{D}\pi \int \mathcal{D}A \exp\left(i\int d^4x \left[\mathcal{L}_{EM} - \frac{1}{2\xi}(\Box\pi)^2\right]\right)$$

shift π : $\pi(x) \longrightarrow \pi(x) - \alpha_A(x)$

$\Box\pi(x) \longrightarrow \Box\pi(x) - \partial_\mu A^\mu(x)$

$$= \int \mathcal{D}\pi \int \mathcal{D}A \exp\left(i\int d^4x \left[\mathcal{L}_{EM} - \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2\right]\right)$$

shift A_μ by a gauge transformation

$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\pi$

$\Box\pi - \partial_\mu A^\mu \longrightarrow \Box\pi - \partial^\mu(A_\mu + \partial_\mu\pi) = -\partial^\mu A_\mu$

$$= \int \mathcal{D}\pi \int \mathcal{D}A \exp\left(i\int d^4x \left[\mathcal{L}_{EM} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]\right)$$

infinite multiplicative constant $\int \mathcal{D}\pi$
cancels between numerator and denominator

LSZ formalism

S-matrix element can be determined from gauge dependent Green function

$$\langle \Omega | T A_{\mu_1}(x_1) A_{\mu_2}(x_2) \cdots | \Omega \rangle$$

S-matrix elements can also be determined from gauge-invariant Green functions

$$\langle \Omega | T F_{\mu_1\nu_1}(x_1) F_{\mu_2\nu_2}(x_2) \cdots | \Omega \rangle$$

with or without gauge fixing term in Lagrangian

S-matrix elements are gauge invariant

$\Longrightarrow$ S-matrix elements can be determined from Green functions of A_μ

with gauge-fixing term in Lagrangian

$$S = \int d^4x \left(\mathcal{L}_{EM} + \mathcal{L}_{gf} \right), \quad \mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$