Path Integral in Quantum Field Theory

quantum field operator $\hat{\phi}(\vec{r}, t=0)$ at different positions commute

$$\left[\hat{\phi}(\vec{r}, t=0), \hat{\phi}(\vec{r}\,', t=0)\right] = 0 \quad \text{for all } \vec{r}, \vec{r}\,'$$

$\Longrightarrow$ the operators $\hat{\phi}(\vec{r}, t=0)$ for all $\vec{r}$
have simultaneous eigenstates
whose eigenvalues define a function $\phi(\vec{r})$

$$\hat{\phi}(\vec{r})\,|\phi\rangle = \phi(\vec{r})\,|\phi\rangle$$

Hamiltonian operator: $\hat{H} = \int d^3r\, \mathcal{H}$

$$\hat{\mathcal{H}} = \frac{1}{2}\dot{\hat{\phi}}^2 + \frac{1}{2}\nabla\hat{\phi}\cdot\nabla\hat{\phi} + \frac{1}{2}m^2\hat{\phi}^2 + V_{int}(\hat{\phi})$$

time evolution operator from time $t=-T$ to time $t=T$

$$\hat{U}(T,-T) = \mathcal{T}\exp\left(-i\int_{-T}^{+T} dt \int d^3x\, \hat{\mathcal{H}}\right)$$

amplitude for state $|\phi_0\rangle$ at time $t=-T$
to evolve into $|\phi_1\rangle$ at time $t=T$:

$$\langle\phi_1|\,U(T,-T)\,|\phi_0\rangle = N \int_{\substack{\phi(\vec{r},-T)=\phi_0(\vec{r}) \\ \phi(\vec{r},T)=\phi_1(\vec{r})}} \mathcal{D}\phi\, \exp\left(i\int d^4x\, \mathcal{L}\right)$$

integrate over paths of fields $\phi(\vec{r},t)$

with initial condition $\phi(\vec{r},-T) = \phi_0(\vec{r})$

final " $\phi(\vec{r},T) = \phi_1(\vec{r})$

Lagrangian density: $\mathcal{L} = \dot{\phi}^2 - \mathcal{H}$

$$= \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}\nabla\phi\cdot\nabla\phi - \tfrac{1}{2}m^2\phi^2 - V_{int}(\phi)$$

$$= \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - V_{int}(\phi)$$

normalization factor: N infinite,

but cancels in ratios

Green function

$$\langle\Omega|T\,\phi(x_1)\cdots\phi(x_n)|\Omega\rangle$$

$$= \frac{\int \mathcal{D}\phi\ \phi(x_1)\cdots\phi(x_n)\exp\left(i\int d^4x\,\mathcal{L}\right)}{\int\mathcal{D}\phi\ \exp\left(i\int d^4x\,\mathcal{L}\right)}$$

$$\int d^4x = \lim_{\epsilon\to 0}\lim_{T\to\infty}\int_{-T(1-\epsilon)}^{+T(1-i\epsilon)} dt \int d^3x$$

Changes of variables

measure $\int \mathcal{D}\phi$ is invariant under shifting field by a function

$$\phi(x) \longrightarrow \phi(x) + f(x)$$

$$\Longrightarrow \int \mathcal{D}\phi \, \exp(iS[\phi]) = \int \mathcal{D}\phi \, \exp(iS[\phi+f])$$

measure $\int \mathcal{D}\phi$ changes by a Jacobian under an invertible linear transformation

$$\phi(x) \longrightarrow \mathcal{O}\,\phi(x)$$

$$\Longrightarrow \int \mathcal{D}\phi \, \exp(iS[\phi]) = \int \mathcal{D}\phi \, (\mathrm{Det}\,\mathcal{O}) \exp(iS[\mathcal{O}\phi])$$

Gaussian integral

$$\int \mathcal{D}\phi \, \exp\left(i \int d^4x \, \phi \, \mathcal{O} \phi\right) = \mathcal{N} \frac{1}{\sqrt{\det \mathcal{O}}}$$

$\mathcal{N}$ = infinite constant

Generating Functional

(for Green function)

$$Z[J] = \int \mathcal{D}\phi \; exp\left(i \int d^4x \, \mathcal{L} + i \int d^4x \, J(x)\,\phi(x) \right)$$

path integral with space-time dependent source $J(x)$
for the field $\phi(x)$

variational derivative with respect to $J(x)$: $\frac{\delta}{\delta J(x)}$

$$\frac{\delta}{\delta J(x)} J(y) = \delta^4(x-y)$$

$$\frac{\delta}{\delta J(x)} \phi(y) = 0$$

$$\frac{\delta}{\delta J(x)} \int d^4x \, \mathcal{L} = 0$$

$$\frac{\delta}{\delta J(x)} \int d^4x \, J(x)\,\phi(x) = \frac{\delta}{\delta J(x)} \int d^4y \, J(y)\,\phi(y)$$

$$= \int d^4y \; \delta^4(x-y)\,\phi(y)$$

$$= \phi(x)$$

variation from infinitesimal change δJ

If $\delta(\quad) = \int d^4x \, W(x)\, \delta J(x)$

then $\frac{\delta}{\delta J(x)}(\quad) = W(x)$

variation of exponent:

$$\delta\, i\int d^4x\left(\mathcal{L}(\phi)+J(x)\phi(x)\right)=i\int d^4x\,\phi(x)\,\delta J(x)$$

$$\frac{\delta}{\delta J(x)}\, i\int d^4x\left(\mathcal{L}(\phi)+J(x)\phi(x)\right)=i\,\phi(x)$$

$$\frac{\delta}{\delta J(x_1)}Z[J]=\int\mathcal{D}\phi\; e^{i\int d^4x(\mathcal{L}+J\phi)}\quad i\,\phi(x_1)$$

$$\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}Z[J]=\int\mathcal{D}\phi\; e^{i\int d^4x(\mathcal{L}+J\phi)}\quad i\,\phi(x_2)\,i\,\phi(x_1)$$

$$\frac{1}{Z[J]}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)Z[J]\Big|_{J=0}$$

$$=\frac{\int\mathcal{D}\phi\; e^{i\int d^4x\,\mathcal{L}}\;\phi(x_1)\phi(x_2)}{\int\mathcal{D}\phi\; e^{i\int d^4x\,\mathcal{L}}}$$

$$=\langle\Omega|T\,\phi(x_1)\phi(x_2)|\Omega\rangle$$

Free real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2$$

$$S[\phi] = \int d^4x\, \mathcal{L} = \int d^4x\, \frac{1}{2}\phi(-\Box - m^2 + i\epsilon)\phi$$

$$= \int d^4x\, \frac{1}{2}\phi\mathcal{O}\phi \qquad \mathcal{O} = -\Box - m^2 + i\epsilon$$

$$Z[J] = \int \mathcal{D}\phi\, \exp\left(i\int d^4x\,[\mathcal{L} + J\phi]\right)$$

$$= \int \mathcal{D}\phi\, \exp\left(i\int d^4x\left[\frac{1}{2}\phi\,\mathcal{O}\phi + J\phi\right]\right)$$

complete the square

$$= \int \mathcal{D}\phi\, \exp\left(i\int d^4x\left[\frac{1}{2}(\phi + J\overleftarrow{\mathcal{O}}^{-1})\mathcal{O}(\phi + \mathcal{O}^{-1}J)\right.\right.$$

$$\left.\left. - \frac{1}{2}J\mathcal{O}^{-1}J\right]\right)$$

shift field: $\phi \to \phi - \mathcal{O}^{-1}J$

$$= \int \mathcal{D}\phi\, \exp\left(i\int d^4x\left[\frac{1}{2}\phi\,\mathcal{O}\phi - \frac{1}{2}J\mathcal{O}^{-1}J\right]\right)$$

$$= \exp\left(-i\int d^4x\,\frac{1}{2}J\,\mathcal{O}^{-1}J\right)\underbrace{\int \mathcal{D}\phi\, \exp\left(i\int d^4x\,\mathcal{L}\right)}_{Z[0]\ \text{(infinite constant)}}$$

$=$

$$Z[J] = Z[0]\, exp\left(-i\int d^4x\, \tfrac{1}{2} J(x)\, \mathcal{O}^{-1} J(x)\right)$$

$$= Z[0]\, exp\left(-i\int d^4x \int d^4x'\, \tfrac{1}{2} J(x)\, \mathcal{O}^{-1}(x,x')\, J(x')\right)$$

$\mathcal{O}^{-1}$ can be expressed as integral transform:

$$\mathcal{O}^{-1} J(x) = \int d^4x'\, \mathcal{O}^{-1}(x,x')\, J(x')$$

$$= \int d^4x' \left(\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} \frac{1}{k^2 - m^2 + i\epsilon} \right) J(x')$$

$$\frac{\delta}{\delta J(x_1)} Z[J] = Z[J]\left(-i\int d^4x \int d^4x'\, \tfrac{1}{2} J(x)\, \mathcal{O}^{-1}(x,x')\, \delta(x'-x_1)\right) \times 2$$

$$= Z[J]\left(-i\int d^4x\, J(x)\, \mathcal{O}^{-1}(x,x_1)\right)$$

$$\frac{\delta}{\delta J(x_2)} \frac{\delta}{\delta J(x_1)} Z[J]\Big|_{J=0} = Z[0]\left(-i\int d^4x\, \delta(x-x_2)\, \mathcal{O}^{-1}(x,x_1)\right)$$

$$= Z[0]\left(-i\, \mathcal{O}(x_2,x_1)\right)$$

$$\langle \Omega | T\phi(x_1)\phi(x_2) | \Omega \rangle = \frac{1}{Z[J]}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0}$$

$$= \frac{1}{Z[0]} (-i)^2\, Z[0] \left(-i\, \mathcal{O}(x_2,x_1)\right)$$

$$= i\, \mathcal{O}(x_1,x_2)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x_1-x_2)} \frac{i}{k^2 - m^2 + i\epsilon}$$