Schwartz Chapter 14: Problems 1*, 2abcd, 3

Problem 14.1* (modified)
(This problem is stated incorrectly in previous versions of Schwartz.)
Derive the following functional integral over a complex scalar field:

$$\int \mathcal{D}\phi^*\mathcal{D}\phi \;\exp\Big(i\textstyle\int d^4x\int d^4y\,\phi^*(x)\,M(x,y)\phi(y) + i\int d^4x\big(J^*(x)\,\phi(x) + \phi^*(x)\,J(x)\big)\Big)$$
$$= \mathcal{N}\frac{1}{\det M}\exp\big(-i\textstyle\int d^4x\int d^4y\,J^*(x)M^{-1}(x,y)J(y)\big),$$

where the integral transform kernel M is hermitian ($M(x,y)^* = M(y,x)$), $J(x)$ is a classical complex source, and $\mathcal{N}$ is an infinite constant.

Problem 14.3 (hints)
(a) Show that the annihilation operator can be expressed as

$$a_{\vec{p}} = \frac{1}{\sqrt{2\omega_p}}\int d^3x\, e^{-i\vec{p}.\vec{x}}\big[\omega_p\hat{\phi}(\vec{x}) + i\hat{\pi}(\vec{x})\big],$$

where $\hat{\phi}(\vec{x}) = \hat{\phi}(\vec{x}, t=0)$ and $\hat{\pi}(\vec{x}) = \partial_t\hat{\phi}(\vec{x}, t=0)$.

(b) Show that $\hat{\pi}(\vec{x})$ acts on eigenstates of $\hat{\phi}$ as the variational derivative $-i\delta/\delta\phi(\vec{x})$ by using the commutation relation

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{x}')] = i\delta(\vec{x} - \vec{x}').$$

(e) To evaluate the integral Eq. (14.66), first go to spherical ccoordinates with the exponential expressed in the form $\exp(ipr\cos\theta)$. After integrating over θ, you will have an integral of the form

$$\int_0^\infty dp\, p\sqrt{p^2+m^2}\,[\exp(ipr) - \exp(-ipr)].$$

You can express this as an integral over the entire real p axis, with only the first exponential in the integrand. You can add to the contour a semicircle at infinity in the upper half plane, and then deform the contour to wrap around the branch cut that runs along the vertical axis from $+im$ to $+i\infty$. You can express this as an integral over y from m to ∞ that Mathematica can evaluate in terms of `BesselK[2, m r]`. If you take the $m \to 0$ limit, you should get a function that is a simple power of r.