Problem 4*

(0) The Lagrangian for \( N \) real scalar fields with an \( O(N) \) symmetry is given in Eq. (23.132). Verify that the Feynman rule for the propagator of a scalar with momentum \( p \) and indices \( i, j \) is

\[
\frac{i\delta^{ij}}{p^2 - m^2 + i\epsilon}.
\]

Verify that the Feynman rule for the 4-scalar vertex with indices \( i, j, k, l \) is

\[
-2i\lambda(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}).
\]

In the case \( N = 1 \), how is the coupling constant \( \lambda \) related to the parameter \( \lambda \) in the Lagrangian in Eq. (23.85).

(a1) Calculate \( \beta(\lambda) \). Check that your result for \( N = 1 \) is consistent with Eq. (23.95).

Calculate the ultraviolet divergent part of each of the three one-loop vertex correction diagrams using dimensional regularization. Determine the renormalization constant \( Z_4 \) for the 4-point vertex to order \( \lambda \) using minimal subtraction. The renormalization constant for \( \lambda \) is \( Z_\lambda = Z_4/(\sqrt{Z_\phi})^4 \). The wavefunction renormalization constant to order \( \lambda \) is \( Z_\phi = 1 \). The relation between the bare and renormalized coupling constants is \( \lambda_0 = \mu^{d-4}Z_\lambda\lambda \). Apply \( \mu d/(d\mu) \) to the logarithm of both sides and use the fact that \( \lambda_0 \) does not depend on \( \mu \).

(a2) Calculate \( \gamma_m(\lambda) \). Check that your result for \( N = 1 \) is consistent with Eq. (23.96).

Treat the \( -\frac{1}{2}m^2\phi^i\phi^j \) term in the Lagrangian as an interaction term whose vertex has two incoming lines and the Feynman rule \( -im^2\delta^{ij} \). Calculate the ultraviolet divergent part of the one-loop vertex correction diagram using dimensional regularization. Determine the renormalization constant \( Z_2 \) for the 2-point vertex to order \( \lambda \) using minimal subtraction. The renormalization counterterm for the mass parameter \( m^2 \) is \( Z_m = Z_2/(\sqrt{Z_\phi})^2 \). The relation between the bare and renormalized masses is \( m_0^2 = Z_mm^2 \). Apply \( \mu d/(d\mu) \) to the logarithm of both sides and use the fact that \( m_0^2 \) does not depend on \( \mu \).
(b) Express the equations for $\beta(\lambda)$ and $\gamma_m(\lambda)$ in the forms in Eqs. (23.104) and (23.105). Identify the nontrivial fixed point analogous to that in Eqs. (23.106).

(c) Predict the critical exponent $\nu$ defined in Eq. (23.103) in $d = 3$ dimensions.