We consider the following functional integral for a complex scalar field \( \phi(x) \) with a complex source \( J(x) \):

\[
\int \mathcal{D}\phi \mathcal{D}\phi^* \exp\left( i \int d^d x \left[ d^d x d^d y \left[ \phi^*(x) - \int d^d w J^*(w) M^{-1}(w,x) \right] \phi(y) \right. \\
+ \left. i \int d^d x \left( \bar{J}(x) \phi(x) + \phi^*(x) J(x) \right) \right] \right)
\]

where the operator \( M \) satisfies \( M(x,y)^* = M(y,x) \).

If we shift the field \( \phi(x) \) by

\[
\phi(x) \rightarrow \phi(x) - \int d^d z M^{-1}(x,z) \phi(z)
\]

the shift in its complex conjugate is

\[
\phi^*(x) \rightarrow \phi^*(x) - \int d^d z \bar{J}^*(z) M^{-1}(z,x)
\]

The functional integral is invariant under the shift. It is therefore equal to

\[
\int \mathcal{D}\phi \mathcal{D}\phi^* \exp\left( i \int d^d x d^d y \left[ \phi^*(x) - \int d^d w J^*(w) M^{-1}(w,x) \right] M(x,y) \left[ \phi(y) - \int d^d z M^{-1}(y,z) \bar{J}(z) \right] \right. \\
+ \left. i \int d^d x \bar{J}^*(x) \left[ \phi(x) - \int d^d z M^{-1}(x,z) J(z) \right] \right) \\
+ i \int d^d x \left[ \phi^*(x) - \int d^d w J^*(w) M^{-1}(w,x) \right] J(x)
\]
\[ = \int D\phi \exp \left( i \int d^4x \left( \partial^\mu \phi - \frac{1}{2} \phi^2 \phi^\mu \right) M(x, y) \phi^\nu (x) - i \int d^4w J^\nu (w) \phi^\nu (w) \right) \]

\[ + i \int d^4x \left( \frac{1}{2} \phi^2 J^\nu (x) - i \int d^4w J^\nu (w) M^{-1}(x, y) J^\nu (y) \right) \]

\[ = \exp \left( -i \int d^4x \left( \frac{1}{2} \phi^2 J^\nu (x) M^{-1}(x, y) J^\nu (y) \right) \right) \]

\[ \times \int D\phi \exp \left( i \int d^4x \left( \partial^\mu \phi - \frac{1}{2} \phi^2 \phi^\mu \right) M(x, y) \phi^\nu (x) \right) \]

The remaining functional integral is a Gaussian integral. It is equal to \(1/\text{det} M\) multiplied by an infinite constant. Thus the complete functional integral is

\[ \sqrt{\text{det} M} \exp \left( -i \int d^4x \left( \frac{1}{2} \phi^2 J^\nu (x) M^{-1}(x, y) J^\nu (y) \right) \right) \]
(a) The Lagrangian for scalar QED is

\[ \mathcal{L} = -\frac{i}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^* D^\mu \phi \]

\[ = -\frac{i}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu - i e A_\mu) \phi^* (\partial^\mu + i e A_\mu) \phi \]

The action of the charge conjugation operator C on the scalar field is

\[ C \phi(x) = \phi^*(x) \]

\[ C \phi^*(x) = \phi(x) \]

For the term \(-i e A_\mu (\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi)\) to be invariant, the action of C on the photon field must be

\[ C A_\mu(x) = - A_\mu(x) \]

(b) The n-photon Green function is

\[ G_{\mu_1 \ldots \mu_n}(x_1, \ldots, x_n) = \langle 0 | T A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) | 0 \rangle \]

\[ = \frac{\int \mathcal{D} A^* \mathcal{D} A e^{i S} A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)}{\int \mathcal{D} A^* \mathcal{D} A e^{i S}} \]
If we make a charge conjugation transformation on the field, the action is invariant and the measure is invariant, but the factors of the photon field in the numerator change

\[ G_{\mu_1 \cdots \mu_n}(x_1, \ldots, x_n) \]

\[ = \frac{\int D\phi^* D\phi \, e^{iS} \, (-A_{\mu_1}(x_1)) \cdots (-A_{\mu_n}(x_n))}{\int D\phi^* D\phi} \]

\[ = (-1)^n \frac{\int D\phi^* D\phi \, e^{iS} \, A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)}{\int D\phi^* D\phi} \]

\[ = (-1)^n \, G_{\mu_1 \cdots \mu_n}(x_1, \ldots, x_n) \]

This implies Furry's Theorem

\[ G_{\mu_1 \cdots \mu_n}(x_1, \ldots, x_n) = 0 \text{ if } n \text{ is odd} \]

(c) It holds when the photons are off-shell. Since the Green function for on-shell photons is obtained by Fourier transforming in all \( n \) coordinates, commutating the propagators, and then putting the 12 momenta on-shell, it also holds if the photons are on-shell.
Schwartz 14.3

(a) The Fourier expansions of the field operator \( \hat{\Phi}(\vec{r}, t) \) and the conjugate momentum operator \( \hat{\Pi}(\vec{r}, t) = \frac{\partial}{\partial t} \hat{\Phi}(\vec{r}, t) \) at time \( t = 0 \) are

\[
\hat{\Phi}(\vec{r}, t = 0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left( \hat{a}_{\vec{p}} e^{-i\vec{p} \cdot \vec{r}} + \hat{a}_{\vec{p}}^+ e^{i\vec{p} \cdot \vec{r}} \right)
\]

\[
\hat{\Pi}(\vec{r}, t = 0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left( -i\omega_p \hat{a}_{\vec{p}} e^{-i\vec{p} \cdot \vec{r}} + (i\omega_p) \hat{a}_{\vec{p}}^+ e^{i\vec{p} \cdot \vec{r}} \right)
\]

The Fourier transforms of these equations are

\[
\int d^3r \ e^{-i\vec{p} \cdot \vec{r}} \ \hat{\Phi}(\vec{r}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \int d^3r_3 \ e^{i\vec{r} \cdot \vec{r}_3} \left( \hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{r}_3} + \hat{a}_{\vec{p}}^+ e^{-i\vec{p} \cdot \vec{r}_3} \right)
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[ \hat{a}_{\vec{p}} \frac{1}{(2\pi)^3} \delta^3(\vec{p} - \vec{r}_3) + \hat{a}_{\vec{p}}^+ \frac{(2\pi)^3}{(2\pi)^3} \delta^3(\vec{p} + \vec{r}_3) \right]
\]

\[
= \frac{1}{\sqrt{2\omega_p}} \left( \hat{a}_{\vec{p}} + \hat{a}_{-\vec{p}}^+ \right)
\]

\[
\int d^3r \ e^{-i\vec{p} \cdot \vec{r}} \ \hat{\Pi}(\vec{r}) = \int \frac{d^3p}{(2\pi)^3} \frac{-i\omega_p}{\sqrt{2\omega_p}} \int d^3r \ e^{-i\vec{p} \cdot \vec{r}} \left( -\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{r}} + \hat{a}_{\vec{p}}^+ e^{-i\vec{p} \cdot \vec{r}} \right)
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{-i\omega_p}{\sqrt{2\omega_p}} \left[ -\hat{a}_{\vec{p}} \frac{1}{(2\pi)^3} \delta^3(\vec{p} - \vec{r}_3) + \hat{a}_{\vec{p}}^+ \frac{(2\pi)^3}{(2\pi)^3} \delta^3(\vec{p} + \vec{r}_3) \right]
\]

\[
= -\frac{i\omega_p}{\sqrt{2\omega_p}} \left( \hat{a}_{\vec{p}} - \hat{a}_{-\vec{p}}^+ \right)
\]

By taking an appropriate linear combination, we can get an expression with only the annihilation operator \( \hat{a}_{\vec{p}} \) on the right side.
\[
\omega_p \int d^3r \ e^{-i \vec{p} \cdot \vec{r}} \hat{\phi}(\vec{r}) + i \int d^3r \ e^{-i \vec{p} \cdot \vec{r}} \hat{\pi}(\vec{r}) = \frac{2\omega_p}{\sqrt{2\omega_p}} \hat{A}_{\phi}
\]

The resulting expression for \( \hat{A}_{\phi} \) is:

\[
\hat{A}_{\phi} = \frac{1}{\sqrt{2\omega_p}} \int d^3r \ e^{-i \vec{p} \cdot \vec{r}} \left( \omega_p \hat{\phi}(\vec{r}) + i \hat{\pi}(\vec{r}) \right)
\]

(b) The operators \( \hat{\phi}(\vec{r}) \) and \( \hat{\pi}(\vec{r}) \) satisfy the canonical commutation relations:

\[
[\hat{\phi}(\vec{r}), \hat{\pi}(\vec{r}')] = i S^3(\vec{r} - \vec{r}')
\]

Suppose \( |\phi\rangle \) is a simultaneous eigenstate of \( \hat{\phi}(\vec{r}) \) for all \( \vec{r} \):

\[
\hat{\phi}(\vec{r}) |\phi\rangle = \phi(\vec{r}) |\phi\rangle
\]

The commutation relation applied to \( |\phi\rangle \) is:

\[
\hat{\pi}(\vec{r}) \hat{\phi}(\vec{r}') |\phi\rangle - \hat{\phi}(\vec{r}) \hat{\pi}(\vec{r}') |\phi\rangle = i S^3(\vec{r} - \vec{r}') |\phi\rangle
\]

We can verify that the action of \( \hat{\pi}(\vec{r}) \) is:

\[
\hat{\pi}(\vec{r}) |\phi\rangle = -i \frac{S^3}{S^3(\vec{r})} |\phi\rangle
\]

The equation can then be written
\[ \Phi(p) \left( -i \frac{\partial}{\partial \Phi(p)} \right) \Phi \rangle = \left( -i \frac{\partial}{\partial \Phi(p)} \right) \Phi \rangle = i S^3(\vec{r}, \vec{r}') \Phi \rangle \]

In the second term, the variational derivative can act either on the factor of \( \Phi(p) \) or on \( \Phi \rangle \). Its action on \( \Phi \rangle \) cancels the first term. Its action on \( \Phi(p) \) is:

\[ \frac{S}{S^3(p)} \Phi(p) = S(\vec{r} - \vec{r}') \]

This gives a term that matches the right side of the equation.

(c) The vacuum state satisfies:

\[ \hat{\mathcal{A}}_p |0\rangle = 0 \quad \text{for all } p \]

The projection onto a simultaneous eigenstate \( |\Phi\rangle \) of \( \hat{\mathcal{A}}_p \) for all \( \vec{r} \) is:

\[ \langle \Phi | \hat{\mathcal{A}}_p | 0 \rangle = 0 \]

Inserting the expression for \( \hat{\mathcal{A}}_p \) in terms of \( \phi(p) \) and \( \pi(p) \), we have:

\[ \frac{1}{\sqrt{2\pi \hbar}} \iint d^3r e^{-i \vec{p} \cdot \vec{r}} \left( \langle \Phi | (\hbar^2 \phi(p) + i \pi(p)) | 0 \rangle = 0 \right) \]
Upon inserting the field representation of $\hat{A}(\vec{r})$ and $\hat{\pi}(\vec{r})$, the becomes
\[
\int d^3r e^{-\gamma \hat{\pi}^2} \left( \omega_0 \phi(\vec{r}) + \frac{S}{S\phi(\vec{r})} \right) \langle \phi|0\rangle = 0
\]

Then inverse Fourier transform gives a functional differential equation for $\langle \phi|0\rangle$:
\[
\frac{S}{S\phi(\vec{r})} \langle \phi|0\rangle = -\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{r}} \left( \int d^3r' e^{-\gamma \hat{\pi}^2} \omega_0 \phi(\vec{r}) \right) \langle \phi|0\rangle
\]
\[
= -\int d^3r' \left( \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} \omega_0 \right) \phi(\vec{r}') \langle \phi|0\rangle
\]

(1) The solution for the vacuum wavefunctional in $E_0$ (14.65) is
\[
\langle \phi|0\rangle = e^{ip \left( -\frac{1}{2} \int d^2x \int d^2y \mathcal{E}(\vec{x},\vec{y}) \phi(\vec{x}) \phi(\vec{y}) \right)}
\]

It's variational derivative is
\[
\frac{S}{S\phi(\vec{r})} \langle \phi|0\rangle = e^{ip} \left( -\frac{1}{2} \int d^2x \int d^2y \mathcal{E}(\vec{x},\vec{y}) \phi(\vec{x}) \phi(\vec{y}) \right) e^{-ip} \phi(\vec{r}) \langle \phi|0\rangle
\]
\[
= -\int d^2r' \mathcal{E}(\vec{r},\vec{r}') \phi(\vec{r}) \langle \phi|0\rangle
\]

The functional differential equation implies that $\mathcal{E}$ is
\[
\mathcal{E}(\vec{r},\vec{r}') = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} \omega_0
\]
The function \( E(\vec{r}, \vec{r}') \) can be written

\[
E(r) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{r}} \frac{1}{\sqrt{p^2 + m^2}}
\]

where \( \vec{r} = \vec{x} - \vec{y} \). Using spherical coordinates for \( \vec{p} \), this becomes

\[
E(r) = \frac{1}{(2\pi)^3} 2\pi \int_0^\infty p \, dp \int_0^{\pi} \sin \theta \, d\theta \, e^{i pr \cos \theta} \frac{1}{\sqrt{p^2 + m^2}}
\]

\[
= \frac{1}{4\pi^2} \int_0^\infty dp \, p^2 \int_{-1}^{+1} e^{i \rho r} \frac{1}{\sqrt{p^2 + m^2}} \left( e^{i \rho r} - e^{-i \rho r} \right)
\]

This can be written as an integral over \( p \) from \(-\infty\) to \(+\infty\)

\[
E(r) = \frac{1}{4\pi^2 i r} \int_{-\infty}^{\infty} dp \, p \sqrt{p^2 + m^2} e^{i \rho r}
\]

As a function of complex \( p \), the integrand decreases exponentially along the positive imaginary axis and it has a square root branch cut running from \( p = +im \) to \(+\infty\). If there was a convergence factor (such as \( 1/(p^2 + m^2) \)), this suppressed the integrand at large \( p \) along the real axis, we could close the contour with a semicircle at \( \infty \) in the upper half plane. We will proceed as if there was such a convergence factor. The integration contour can then be deformed to wrap around the branch cut running from \(+\infty - \epsilon\) to \(i m\) to \(+\infty + \epsilon\)
Parameterize the contour by \( p = i \gamma y \), where \( y \) runs from \( \infty + i \epsilon \) to \( 1 \) to \( \infty - i \epsilon \). The function becomes

\[
E(r) = \frac{1}{4\pi^2 i r} \left( \int_{i\infty}^{1} \imath \gamma dy \, \imath \gamma y \, m^2 \sqrt{-(y-\imath \epsilon)^2 + 1} \, e^{-\imath \gamma r y} \right)

+ \int_{1}^{\infty} \imath \gamma dy \, \imath \gamma y \, m^2 \sqrt{-(y-\imath \epsilon)^2 + 1} \, e^{-\imath \gamma r y}

= \frac{1}{4\pi^2 i r} \left( -m^4 \right) \int_{0}^{\infty} dy \, y \left( e^{i \gamma y \sqrt{y^2 - 1}} - e^{-i \gamma y \sqrt{y^2 - 1}} \right) e^{-\imath \gamma r y}

= -\frac{m^4}{4\pi^2 i r} \left( -2i \right) \int_{0}^{\infty} dy \, y \sqrt{y^2 - 1} e^{-\imath \gamma r y}

= -\frac{m^4}{2\pi^2 r} \cdot \frac{1}{mr} K_2(mr)

where \( K_2(z) \) is a Bessel function. Its limiting behavior at small \( m \) is

\[ K_2(mr) \to \frac{2}{(mr)^3} \]

Thus the limit of the function as \( m \to 0 \) is

\[ E(r) \to -\frac{1}{\pi^2 r^5} \]