Schwartz 14.5

(a) To get an equation for $\Box^{\mu\nu}\langle A_\nu A_\alpha \phi^* \phi\rangle$, we begin with the path integral expression for the Green function $\langle A_\alpha \phi^* \phi\rangle$

$$\langle A_\alpha(y)\, \phi^*(z)\, \phi(w)\rangle = \frac{\int \mathcal{D}A\mathcal{D}\phi^*\mathcal{D}\phi\, A_\alpha(y)\,\phi^*(z)\,\phi(w)\, e^{i\int d^4x\,\mathcal{L}}}{\int \mathcal{D}A\mathcal{D}\phi^*\mathcal{D}\phi\, e^{i\int d^4x\,\mathcal{L}}}$$

where the Lagrangean is

$$\mathcal{L} = (\partial_\mu \phi^* - ieA_\mu \phi^*)(\partial_\mu \phi + ieA_\mu \phi)$$

$$- \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A_\mu) - \tfrac{1}{2\xi}(\partial_\mu A^\mu)^2$$

The path integral is invariant under shifting the photon field by a function

$$A_\mu(x) \longrightarrow A_\mu(x) + a_\mu(x)$$

The first order change in the action is

$$\delta S = \int d^4x \Big(-ie\, a_\mu \phi^* \partial^\mu \phi + ie\, a_\mu \partial^\mu \phi^* \phi \;+ e^2 \phi^*\phi\, 2A^\mu a_\mu$$

$$-\tfrac{1}{4}\cdot 2(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu a_\nu - \partial_\nu a_\mu) \;- \tfrac{1}{2\xi} 2(\partial_\mu A^\mu)\, \partial_\nu a^\nu\Big)$$

We can integrate by parts and interchange indices to get a overall multiplicative factor a_μ inside the integral:

$$\delta S = \int d^4x\, a_\mu \Big(-ie\,(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) + 2e^2 A^\mu \phi^* \phi$$

$$+ (g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu) A_\nu + \tfrac{1}{\xi} \partial^\mu \partial^\nu A_\nu \Big)$$

The first order change in the numerator of the path integral is

$$\int \mathcal{D}A \mathcal{D}\phi^* \mathcal{D}\phi\, e^{i\int d^4x \mathcal{L}} \left[a_\alpha(y) \phi^*(z) \phi(w) + A_\alpha(y) \phi^*(z) \phi(w)\, i\delta S \right]$$

$$= i \int \mathcal{D}A \mathcal{D}\phi^* \mathcal{D}\phi\, e^{i\int d^4x \mathcal{L}} \Big(-i \int d^4x\, a_\mu(x)\, g^\mu{}_\alpha\, \delta^4(x-y)\, \phi^*(z) \phi(w)$$

$$+ A_\alpha(y) \phi^*(z) \phi(w)\, \delta S \Big]$$

For the first order change to be zero, the coefficient of $\int d^4x\, a_\mu(x)$ must be 0:

$$\int \mathcal{D}A \mathcal{D}\phi^* \mathcal{D}\phi\, e^{i\int d^4x \mathcal{L}} \Big(-i\, \delta^4(x-y)\, g^\mu{}_\alpha\, \phi^*(z) \phi(w)$$

$$+ A_\alpha(y) \phi^*(z) \phi(w) \Big[-ie \left(\phi^*(x) \partial^\mu \phi(x) - \phi(x) \partial^\mu \phi^*(x) \right) + 2e^2 A^\mu(x)$$

$$+ \left(g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu + \tfrac{1}{\xi} \partial^\mu \partial^\nu \right) A_\nu(x) \Big] = 0$$

Dividing by the normalizing path integral, we can read off the Schwinger-Dyson equation:

$$\boxed{\begin{aligned} &\Box_x^{\mu\nu} \langle A_\nu(x) A_\alpha(y) \phi^*(z) \phi(w) \rangle + 2e^2 \langle A^\mu(x) A_\nu(y) \phi^*(z) \phi(w) \rangle \\ &- ie \langle (\phi^*(x) \partial^\mu \phi(x) - \phi(x) \partial^\mu \phi^*(x)) A_\nu(y) \phi^*(z) \phi(w) \rangle - i\, g^\mu{}_\alpha\, \delta^4(x-y) \langle \phi^*(z) \phi(w) \rangle = 0 \end{aligned}}$$

(b) To get the current conservation Schwinger–Dyson equation for scalar QED, we start with the path integral

$$\int \mathcal{D}A \int \mathcal{D}\phi^* \mathcal{D}\phi \; \exp(iS) \, A_\alpha(y) \, \phi(z) \, \phi^*(w)$$

This is invariant under the change of variables by a local phase transformation:

$$\phi(x) \longrightarrow e^{+i\epsilon(x)} \phi(x)$$
$$\phi^*(x) \longrightarrow e^{-i\epsilon(x)} \phi^*(x)$$
$$A_\mu(x) \longrightarrow A_\mu(x)$$

The infinitesimal changes in the fields are

$$\delta\phi(x) = +i\epsilon(x)\,\phi(x)$$
$$\delta\phi^*(x) = -i\,\epsilon(x)\,\phi^*(x)$$
$$\delta A_\mu(x) = 0$$

The infinitesimal change in the action comes from terms with derivatives acting on factors of $\epsilon(x)$ in variation of ϕ and ϕ^*:

$$\delta S = \int d^4x \; \delta\left(\partial^\mu\phi^* \partial_\mu \phi - ie\, A^\mu \phi^* \partial_\mu \phi + ie A_\mu \partial^\mu \phi^* \phi \right)$$

$$= \int d^4x \Big[(-i\,\partial^\mu\epsilon\,\phi^*)\partial_\mu\phi + \partial^\mu\phi^*(+i\partial_\mu\epsilon\,\phi)$$

$$-ie\, A^\mu\phi^*(+i\partial^\mu\epsilon\,\phi) + ieA_\mu(-i\partial^\mu\epsilon)\phi^*\phi \Big]$$

We can integrate by parts to get a factor of $\epsilon(x)$ without any derivative in the integrand

$$\delta S = \int d^4x\, \epsilon(x) \Big[i\partial^\mu(\phi^* \partial_\mu \phi) - i\partial_\mu(\partial^\mu \phi^* \phi)$$

$$+ 2e\, \partial^\mu (A_\mu \phi^* \phi) \Big]$$

The infinitesimal changes in the product of field in the path integral is

$$\delta\Big(A_\alpha(y)\, \phi(z)\, \phi^*(w)\Big) = A_\alpha(y) \Big[\delta\phi(z)\, \phi^*(w) + \phi(z)\, \delta\phi^*(w) \Big]$$

$$= A_\alpha(y) \Big[\big(i\epsilon(z)\phi(z)\big)\phi^*(w) + \phi(z)\big(-i\epsilon(w)\phi^*(w)\big) \Big]$$

We can use delta function to express this as an integral over x with a factor of $\epsilon(x)$ in the integrand:

$$\delta\Big(A_\alpha(y)\, \phi(z)\, \phi^*(w)\Big) = \int d^4x\, \epsilon(x)\, A_\alpha(y) \Big[i\delta(z-x)\, \phi(x)\, \phi^*(w)$$

$$- i\,\delta(w-x)\, \phi(z)\, \phi^*(x) \Big]$$

The infinitesimal change in the path integral must be 0:

$$0 = \int \mathcal{D}A \int \mathcal{D}\phi^* \mathcal{D}\phi\, e^{iS} \Big[i\delta S\, A_\alpha(y)\, \phi(z)\, \phi^*(w)$$

$$+ \delta\Big(A_\alpha(y)\, \phi(z)\, \phi^*(w)\Big) \Big]$$

It can be expressed as an integral over x with a factor of $\epsilon(x)$ in the integrand:

$$0 = \int d^4x\, \epsilon(x) \int \mathcal{D}A \int \mathcal{D}\phi^* \mathcal{D}\phi\, e^{iS}$$

$$\times \Big[\frac{\partial}{\partial x^\mu}\left(i\phi^* \partial^\mu \phi - i\partial^\mu \phi^* \phi + 2eA^\mu \phi^* \phi \right) A_\alpha(y)\, \phi(z)\, \phi^*(w)$$

$$+ A_\mu(y) \left(i\delta(z-x)\, \phi(x)\, \phi^*(w) - i\delta(w-x)\, \phi(z)\, \phi^*(x) \right) \Big]$$

Since the must be 0 for all functions $\epsilon(x)$, the function multiplying $\epsilon(x)$ in the integrand must be 0. After dividing each term by the unweighted path integral, we get an equation for correlation functions:

$$\frac{\partial}{\partial x^\mu} \left\langle \left(i\phi^* \partial^\mu \phi(x) - i\partial^\mu \phi^* \phi(x) + 2eA^\mu \phi^* \phi(x) \right) A_\alpha(y)\, \phi(z)\, \phi^*(w) \right\rangle$$

$$+ i\,\delta(z-x) \left\langle A_\mu(y)\, \phi(x)\, \phi^*(w) \right\rangle$$

$$- i\,\delta(w-x) \left\langle A_\mu(y)\, \phi(z)\, \phi^*(x) \right\rangle = 0$$

The correlation function in the first term has the form $\langle j^\mu(x)\, A_\alpha(y)\, \phi(z)\, \phi^*(w) \rangle$, where the current is

$$j^\mu = i\phi^* \partial^\mu \phi - i\partial^\mu \phi^* \phi + 2eA^\mu \phi^* \phi$$

$$= i\phi^* D^\mu \phi - i D^\mu \phi^* \phi$$

It differs from the current in the free theory by the term $2eA^\mu \phi^* \phi$.

Schwartz 14.6

(a) If we make the spinor indices on the fermion field ψ explicit, the term $(\bar{\psi}(x)\psi(x))^2$ becomes

$$(\bar{\psi}(x)\psi(x))^2 = \sum_{a,b=1}^{4} \psi_a^\dagger(x)(\gamma_0)_{ab}\psi_b(x) \sum_{c,d=1}^{4} \psi_c^\dagger(x)(\gamma_0)_{cd}\psi_d(x)$$

Since ψ is a complex spinor field, $\psi_a(x)$ and $\psi_a^\dagger(x)$ are independent Grassman fields. An alternative choice of independent fields is $\psi_a(x)$ and $\bar{\psi}_a(x)$. This term can therefore be written

$$(\bar{\psi}(x)\psi(x))^2 = \sum_{a=1}^{4} \bar{\psi}_a(x)\psi_a(x) \cdot \sum_{b=1}^{4} \bar{\psi}_b(x)\psi_b(x)$$

Using the fact that Grassmann fields anticommute and that $\psi_a(x)^2 = 0$ and $\bar{\psi}_a(x)^2 = 0$, the term can be written as a sum over distinct pairs of indices:

$$(\bar{\psi}(x)\psi(x))^2 = 2\sum_{1\le a<b\le 4} \bar{\psi}_a(x)\psi_a(x)\bar{\psi}_b(x)\psi_b(x)$$

This is nonzero. On the other hand, $(\bar{\psi}(x)\psi(x))^5$ is zero, because each of the 4^5 terms has two factors that are the same Grassman field $\psi_a(x)$.

If $(\bar{\Psi}(x)\Psi(x))^5$ was nonzero, such a term in the Lagrangean could contribute at tree level to the reaction $e^+e^- \longrightarrow 4(e^+e^-)$. However this term is 0 in the path integral formalism.

In the operator formalism, the fields satisfy the equal-time anticommutation relations

$$\{\Psi_a(\vec{r},t), \Psi_b(\vec{r}',t)\} = 0$$

$$\{\Psi_a(\vec{r},t), \bar{\Psi}_b(\vec{r}',t)\} = i(\gamma_0)_{ab}\,\delta^3(\vec{r}-\vec{r}')$$

$$\{\bar{\Psi}_a(\vec{r},t), \bar{\Psi}_b(\vec{r}',t)\} = 0$$

Setting $\vec{r}'=r$, the first and third anticommutation relations imply

$$\Psi_a(x)^2 = 0 = \bar{\Psi}_a(x)^2 = 0$$

In the operator $(\bar{\Psi}(x)\Psi(x))^5$, moving the repeated operators $\Psi_a(x)$ so that they are adjacent and therefore vanish requires using the second anticommutation relation. Thus the operator $(\bar{\Psi}(x)\Psi(x))^5$ can be reduced to terms of the form $i\,\delta^3(0)\,(\bar{\Psi}(x)\Psi(x))^4$. It therefore cannot contribute at tree level to the reaction $e^+e^- \rightarrow 4(e^+e^-)$.

(b). The gauge-fixing term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ can be generalized to a function $-\frac{1}{2\xi}[f(\partial_\mu A^\mu)]^2$. Such a term can be introduced by starting with the path integral without a gauge-fixing term but an additional factor

$$\int \mathcal{D}\pi \, e^{-\frac{1}{2\xi}\int d^4x \, f(\Box\pi)^2}$$

If we shift the field π by $\pi \to \pi + \frac{1}{\Box}\partial_\mu A^\mu$, the change in the integrand is

$$f(\Box\pi) \longrightarrow f(\Box\pi + \partial_\mu A^\mu)$$

If we then make a gauge transformation $A_\mu \to A_\mu - \partial_\mu \pi$, the change in the integrand is

$$f(\Box\pi + \partial_\mu A^\mu) \longrightarrow f(\partial_\mu A^\mu)$$

This is the generalized gauge-fixing term. The gauge-fixing term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^4$ is obtained by choosing $f(\partial_\mu A^\mu) = (\partial_\mu A^\mu)^2$.

The term $\xi A_\mu A^\mu$ cannot be obtained as a gauge-fixing term by this method. To see that it is not a reasonable gauge fixing term, take the limit $\xi \to \infty$. Such a term in the Lagrangian of the path integral would imply $A_\mu(x) = 0$ for all x. Instead of eliminating the gauge freedom, it eliminate the gauge field altogether.