Schwartz 14.5

(a) To get an equation for $\Delta^{\mu
u}\langle A_\alpha A_\delta \phi^\ast \phi \rangle$, we begin with the path integral expression for the Green function $\langle A_\alpha \phi^\ast \phi \rangle$

$$\langle A_\alpha (y) \phi^\ast (z) \phi (w) \rangle = \frac{\int [DA_\alpha D\phi^\ast D\phi] A_\alpha (y) \phi^\ast (z) \phi (w) e^{iS_{D\phi^\ast D\phi} / \hbar}}{\int [DA_\alpha D\phi^\ast D\phi] e^{iS_{D\phi^\ast D\phi} / \hbar}}$$

where the Lagrangian is

$$L = (\partial_\mu \phi^\ast - ieA_\mu \phi^\ast)(\partial^\mu \phi + ieA_\mu \phi)$$

$$- \frac{i}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{4\pi} (\partial_\mu A^\mu)^2$$

The path integral is invariant under shifting the photon field by a function

$$A_\mu (x) \rightarrow A_\mu (x) + A_\mu (x)$$

The first order change in the action is

$$\delta S = \int d^4x \left( -ie \phi^\ast \phi^\ast \partial_\mu \phi + ie \phi^\ast \phi \partial_\mu \phi^\ast + e^2 \phi^\ast \phi 2A_\mu \partial_\mu \phi^\ast \right.$$

$$\left. - \frac{i}{4} 2 (\partial_\mu A^\nu - \partial_\nu A^\mu)(\partial_\mu A^\nu - \partial_\nu A^\mu) - \frac{1}{2\pi} 2(\partial_\mu A^\mu) \partial_\nu A^\nu \right)$$

We can integrate by parts and interchange indices to get an overall multiplicative factor $A_\mu$ inside the integral.
\[ SS = \int d^4 x \ g_\mu \left( -i e \left( \phi^* \partial^\mu \phi - \phi \partial^\mu \phi^* \right) + 2 e^2 A^\mu \phi^* \phi \right) + (g^{\mu \nu} \partial^\mu \partial^\nu - \partial^\rho \partial_\rho) A_\nu + \frac{1}{3} \partial^\mu \partial^\nu A_\mu \right) \]

The first order change in the numerator of the path integral is
\[ \int DO \phi^* \partial \phi = i \int d^4 x \left[ A_\alpha (y) \phi^* (z) \phi (w) + A_\alpha (y) \phi^* (z) \phi (w) \right] \]

For the first order change to be zero, the coefficient of \( \int d^4 x \ g_\mu \) must be 0:
\[ \int DO \phi^* \partial \phi = i \int d^4 x \left( -i S''(x-y) g^\mu \phi^* (z) \phi (w) \right) \]

\[ + A_\alpha (y) \phi^* (z) \phi (w) \left[ -i e \left( \phi^* (x) \partial^\mu \phi (x) - \phi (x) \partial^\mu \phi^* (x) \right) + 2 e^2 A^\mu (x) \right] \]

\[ + (g^{\mu \nu} \partial^\mu \partial^\nu + \frac{1}{3} \partial^\mu \partial^\nu) A_\nu (x) \right] = 0 \]

Dividing by the normalizing path integral, we can read off the Schwinger-Dyson equation:
\[ \square^\mu \left( A_\nu (x) A_\alpha (y) \phi^* (z) \phi (w) \right) + 2 e^2 \left( A^\mu (x) A_\nu (y) \phi^* (z) \phi (w) \right) \]

\[ - i e \left( \phi^* (x) \partial^\mu \phi (x) - \phi (x) \partial^\mu \phi^* (x) \right) A_\nu (y) \phi^* (z) \phi (w) \right) - i g^\mu \phi^* S'' (x-y) \phi^* (z) \phi (w) = 0 \]
(b) To get the current conservation Schwinger-Dyson equation for scalar QED, we start with the path integral

$$\int D\Phi \exp(iS) A(x) \Phi(z) \Phi^*(w)$$

This is invariant under the change of variables by a local phase transformation:

$$\Phi(x) \rightarrow e^{\pm i \epsilon(x)} \Phi(x)$$
$$\Phi^*(x) \rightarrow e^{-i \epsilon(x)} \Phi^*(x)$$
$$A\mu(x) \rightarrow A\mu(x)$$

The infinitesimal changes in the fields are

$$S \Phi(x) = +i \epsilon(x) \Phi(x)$$
$$S\Phi^*(x) = -i \epsilon(x) \Phi^*(x)$$
$$SA\mu(x) = 0$$

The infinitesimal change in the action comes from terms with derivative acting on factors of $\epsilon(x)$ in variations of $\Phi$ and $\Phi^*$:

$$SS = \int d^4x \left[ 2\Phi^* \mathcal{D}_{\mu} \Phi - i e A^\mu \Phi^* \mathcal{D}_{\mu} \Phi + i e A^\mu \mathcal{D}_{\mu} \Phi^* \Phi \right]$$

$$= \int d^4x \left[ -i D^\mu \epsilon \Phi^* \right] \mathcal{D}_{\mu} \Phi + D^\mu \Phi^* (i e \epsilon \Phi)$$

$$-i e A^\mu \Phi^* (i e \epsilon \Phi) + i e A^\mu \left( -i e \epsilon \Phi \right)^* \Phi$$
We can integrate by parts to get a factor of \( e(x) \) without any derivative in the integrand:

\[
SS = \int d^n x \ e(x) \left[ i \partial^\mu (\phi^* \partial_{\mu} \phi) - i \partial_{\mu} (\partial^\mu \phi^* \phi) \right]
+ 2e \int d^n \left( A_\mu \phi^* \phi \right)
\]

The infinitesimal change in the product of fields in the path integral is

\[
S(A_\alpha(y) \phi(z) \phi^*(w)) = A_\alpha(y) \left[ S(\phi(z) \phi^*(w)) + \phi(z) S(\phi^*(w)) \right]
= A_\alpha(y) \left[ i \epsilon (z) \phi(z) \phi^*(w) + \phi(z) (-i \epsilon (w) \phi^*(w)) \right]
\]

We can use delta function to express the as an integral over \( x \) with a factor of \( e(x) \) in the integrand:

\[
S(A_\alpha(y) \phi(z) \phi^*(w)) = \int d^n x \ e(x) A_\alpha(y) \left[ i S(z-x) \phi(x) \phi^*(w) - i S(w-x) \phi(z) \phi^*(w) \right]
\]

The infinitesimal change in the path integral must be 0:

\[
0 = \int dA_\mu d^4 \phi d^4 \phi^* e^{i S} \left[ i SS A_\alpha(y) \phi(z) \phi^*(w) + S(A_\alpha(y) \phi(z) \phi^*(w)) \right]
\]
It can be expressed as an integral over \( x \) with a factor of \( e(x) \) in the integrand:

\[
0 = \int d^4 x \, e(x) \int d\Omega \, d\phi \, \phi \, e^{iS}
\]

\[
\times \left[ \frac{\partial}{\partial x^\mu} \left( i \phi^* \partial_\mu \phi - i \phi \partial_\mu \phi^* + 2e A^\mu \phi \phi^* \right) A_\nu(y) \phi(z) \phi^*(w) \\
+ A_\mu(y) \left( iS(z-x) \phi(x) \phi^*(w) - iS(w-x) \phi(z) \phi^*(x) \right) \right]
\]

Since the must be 0 for all functions \( e(x) \), the function multiplying \( e(x) \) in the integrand must be 0. After dividing each term by the unweighted path integral, we get an equation for correlation functions:

\[
\frac{\partial}{\partial x^\mu} \left< \left( i \phi^* \partial_\mu \phi - i \phi \partial_\mu \phi^* + 2e A^\mu \phi \phi^* \right) A_\nu(y) \phi(z) \phi^*(w) \right> \\
+ iS(z-x) \left< A_\mu(y) \phi(x) \phi^*(w) \right> \\
- iS(w-x) \left< A_\mu(y) \phi(z) \phi^*(x) \right> = 0
\]

The correlation function in the first term has the form \( \left< j^\mu(x) A_\nu(y) \phi(z) \phi^*(w) \right> \), where the current is

\[
\begin{align*}
    j^\mu &= i \phi^* \partial^\mu \phi - i \phi \partial^\mu \phi^* + 2e A^\mu \phi \phi^* \\
    &= i \phi^* D^\mu \phi - i D^\mu \phi^* \phi
\end{align*}
\]

It differs from the current in the free theory by the term \( 2e A^\mu \phi \phi^* \phi \).
Schwartz 14.6

(a) If we make the spinor indices on the fermion field $\psi^a(x)$ explicit, the term $(\overline{\psi}(x)\psi(x))^2$ becomes

$$\overline{\psi}(x)\psi(x)^2 = \sum_{a,b=1}^4 \overline{\psi}^a(x)(\gamma_0)_{ab} \psi^b(x) \sum_{c,d=1}^4 \overline{\psi}^c(x)(\gamma_0)_{cd} \psi^d(x)$$

Since $\psi$ is a complex spinor field, $\psi^a(x)$ and $\overline{\psi}^a(x)$ are independent Grassman fields. An alternative choice of independent fields is $\psi^a(x)$ and $\overline{\psi}^a(x)$. This term can therefore be written

$$\overline{\psi}(x)\psi(x)^2 = \sum_{a=1}^4 \overline{\psi}^a(x)\psi^a(x) \cdot \sum_{b=1}^4 \overline{\psi}^b(x)\psi^b(x)$$

Using the fact that Grassmann fields anticommute and that $\psi^a(x)^2 = 0$ and $\overline{\psi}^a(x)^2 = 0$, the term can be written as a sum over distinct pairs of indices:

$$\overline{\psi}(x)\psi(x)^2 = 2 \sum_{1 \leq a < b \leq 4} \overline{\psi}^a(x)\psi^a(x)\overline{\psi}^b(x)\psi^b(x)$$

This is nonzero. On the other hand, $(\overline{\psi}(x)\psi(x))^5$ is zero, because each of the 45 terms has two factors that are the same Grassman field $\psi^a(x)$. 
If \((\bar{\Psi}(x)\Psi(x))^5\) was non-zero, such a term in the Lagrangian could contribute at tree level to the reaction \(e^+e^- \rightarrow 4(e^+e^-)\). However, this term is 0 in the path-integral formalism.

In the operator formalism, the fields satisfy the equal-time anticommutation relations:

\[
\{ \Psi_a(\vec{r},t), \Psi_b(\vec{r}',t) \} = 0
\]

\[
\{ \Psi_a(\vec{r},t), \bar{\Psi}_b(\vec{r}',t) \} = i (\gamma_\alpha)_{ab} S^3(\vec{r} - \vec{r}')
\]

\[
\{ \bar{\Psi}_a(\vec{r},t), \bar{\Psi}_b(\vec{r}',t) \} = 0
\]

Setting \(\vec{r}' = \vec{r}\), the first and third anticommutation relations imply

\[
\Psi_a(x)^2 = 0 = \bar{\Psi}_a(x)^2 = 0
\]

In the operator \((\bar{\Psi}(x)\Psi(x))^5\), moving the repeated operator \(\Psi_a(x)\) so that they are adjacent and therefore vanish requires using the second anticommutation relation. Thus the operator \((\bar{\Psi}(x)\Psi(x))^5\) can be reduced to terms of the form \(iS^3(0)(\bar{\Psi}(x)\Psi(x))^4\). It therefore cannot contribute at tree level to the reaction \(e^+e^- \rightarrow 4(e^+e^-)\).
(b). The gauge-fixing term \(-\frac{1}{2\pi} \left( \partial \pi \right)^2\) can be
generalized to a function \(-\frac{1}{2\pi} \left[ S(\partial \pi) \right]^2\).
Such a term can be introduced by starting
with the path integral without a gauge-fixing
term but an additional factor
\[
\int D\pi e^{-\frac{1}{2\pi} \left[ S(\partial \pi) \right]^2}
\]
If we shift the field \(\pi\) by \(\pi \rightarrow \pi + \frac{1}{\partial A^\mu}\),
the change in the integrand is
\[
S(\partial \pi) \rightarrow S(\partial \pi + \partial A^\mu)
\]
If we then make a gauge transformation \(A_\mu \rightarrow A_\mu - \partial_\mu \pi\),
the change in the integrand is
\[
S(\partial \pi + \partial A^\mu) \rightarrow S(\partial A^\mu)
\]
This is the generalized gauge-fixing term. The
gauge-fixing term \(-\frac{1}{2\pi} \left( \partial \pi \right)^2\) is obtained by
choosing \(S(\partial \pi) = (\partial A^\mu)^2\).

The term \(\partial A^\mu\) cannot be obtained as a gauge-fixing
term by this method. To see that it is not a reasonable
gauge-fixing term, take the limit \(\delta \rightarrow 0\). Such a term
in the Lagrangian of the path integral would simply
\(A_\mu(x) = 0\) for all \(x\). Instead of eliminating the gauge
freedom, it eliminates the gauge-field altogether.