Goldenfeld, Chapter 2

Exercise 2-1

(a) We consider the Ising model with temperature parameter $\beta = \alpha_1 \beta_1 + \alpha_2 \beta_2$, where $\alpha_1 + \alpha_2 = 1$. The partition function is

$$Z(\alpha_1\beta_1 + \alpha_2\beta_2) = \mathrm{Tr}\, e^{-(\alpha_1\beta_1+\alpha_2\beta_2)H}$$

$$= \sum_k e^{-(\alpha_1\beta_1+\alpha_2\beta_2)E_k}$$

$$= \sum_k \left(e^{-\beta_1 E_k}\right)^{\alpha_1}\left(e^{-\beta_2 E_k}\right)^{\alpha_2}$$

The Holder inequality implies

$$Z(\alpha_1\beta_1 + \alpha_2\beta_2) \leq \left(\sum_k e^{-\beta_1 E_k}\right)^{\alpha_1}\left(\sum_k e^{-\beta_2 E_k}\right)^{\alpha_2}$$

$$= Z(\beta_1)^{\alpha_1} Z(\beta_2)^{\alpha_2}$$

Taking the logarithm of both sides, we get

$$\log Z(\alpha_1\beta_1+\alpha_2\beta_2) \leq \alpha_1 \log Z(\beta_1) + \alpha_2 \log Z(\beta_2)$$

Thus $\log Z$ is convex down in β. The function $g(\beta)$ is obtained by dividing by N and taking the thermodynamic limit $N \to \infty$. Therefore $g(\beta)$ is also convex down.

(b) The free energy per site is

$$f(T) = -kT\, g(\beta), \quad \text{where } \beta = \frac{1}{kT}$$

Its first derivative is

$$f'(T) = -kT\, g'(\beta)\left(-\frac{1}{kT^2}\right) - k\, g(\beta)$$

$$= \frac{1}{T} g'(\beta) - k\, g(\beta)$$

Its second derivative is

$$f''(T) = \frac{1}{T} g''(\beta)\left(-\frac{1}{kT^2}\right) - \frac{1}{T^2} g'(\beta) - k\, g'(\beta)\left(-\frac{1}{kT^2}\right)$$

$$= -\frac{1}{kT^3} g''(\beta)$$

Wherever $g'(\beta)$ is smooth, $g''(\beta) \geq 0$, so $f''(T) \leq 0$

If $g'(\beta)$ has a discontinuity at β_0, it must satisfy $g'(\beta_0^+) > g'(\beta_0^-)$. Therefore $f'(T)$ must satisfy $f'(T_0^-) > f'(T_0^+)$ or equivalently $f'(T_0^+) < f'(T_0^-)$

Thus $f'(T)$ is a nonincreasing function of T
This is consistent with $f(T)$ being concave up in T.

(c) The Gibbs free energy is

$$\Gamma(M) = F(H(M)) + NMH(M)$$

where $H(M)$ is a solution to $M(H) = -\frac{1}{N}F'(H)$
The first derivative of Γ is

$$\Gamma'(M) = F'(H(M))\,H'(M) + NH(M) + NMH'(M)$$

$$= [F'(H(M)) + NM]\,H'(M) + NH(M)$$

$$= NH(M)$$

The second derivative of Γ is

$$\Gamma''(M) = NH'(M)$$

$$= N\frac{1}{M'(H(M))}$$

The derivative of M is

$$M'(H) = -\frac{1}{N}F''(H)$$

Thus the second derivative of Γ is

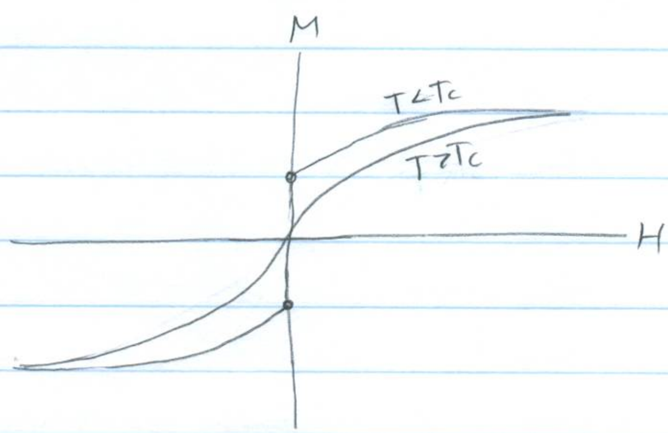
$$\Gamma''(M) = -\frac{N^2}{F''(H(M))}$$

Since F is concave up in H, $F''(H) \leq 0$ almost everywhere. Therefore $\Gamma''(M) \geq 0$ almost everywhere

At the exceptional points H_0 where $F'(H)$ is discontinuous, it decreases between H_0^- and H_0^+. The expression $\Gamma'(M) = NH(M)$ implies that $\Gamma'(M)$ is continuous at that point

Thus $\Gamma'(M)$ is nondecreasing. This is consistent with it being a concave down function of M.

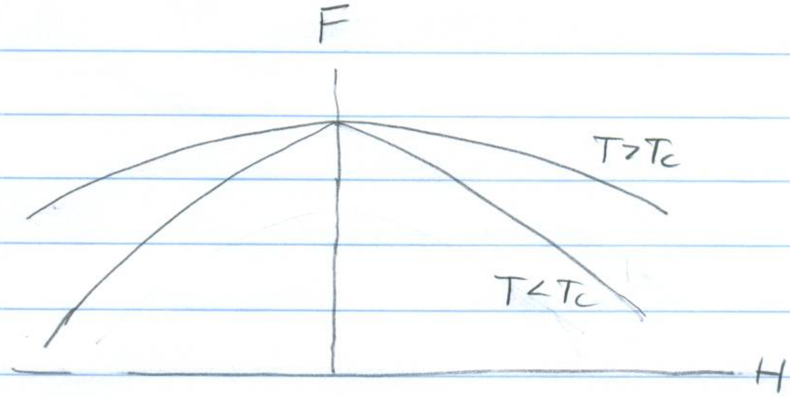
(d) The behavior of $M(H)$ is



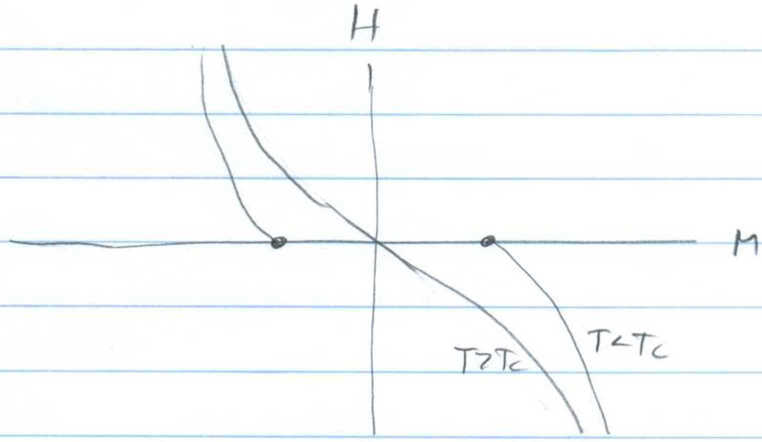
Since $M = -\frac{1}{N}F'(H)$, F can be obtained by integration:

$$F(H) = F(0) - N\int_0^H dH\, M(H)$$

Thus the behavior of $F(H)$ is

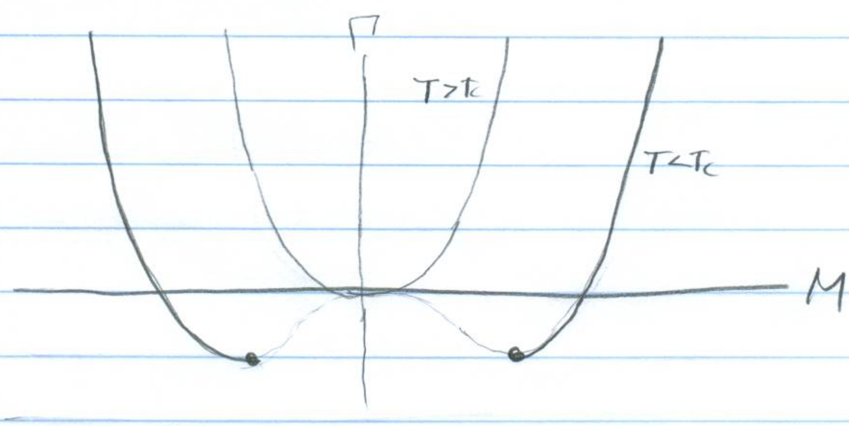


The behavior of H(M) can be obtained from the graph of M(H) by interchanging the axes

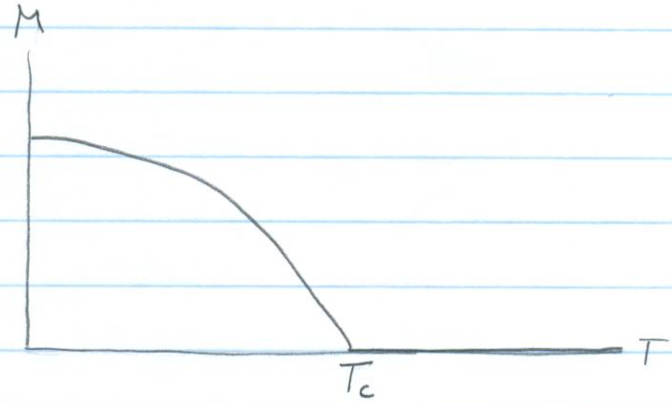


$\Gamma(M)$ can be obtained from H(M) by integration:

$$\Gamma(M) = \Gamma(0) + N\int_0^M dM\, H(M)$$



At $H=0^+$, the behavior of $M(T)$ is



Goldenfeld, Chapter 2

Exercise 2-2

(a) The Hamilton should be extensive: it should scale as the number N of sites. The first term is extensive because the sum is over the N sites. In the second term, the sum is over the N^2 pairs of sites. For it to be extensive, the coefficient J_0 must scale as $1/N$.

(b) The integral is convergent if the real part of a is positive. The integral is

$$\int_{-\infty}^{\infty} dy\, e^{-\frac{N}{2}ay^2 + axy} = \int_{-\infty}^{\infty} dy\, e^{-\frac{N}{2}a(y - x/N)^2 + ax^2/2N}$$

$$= e^{ax^2/2N} \int_{-\infty}^{\infty} dy'\, e^{-\frac{N}{2}a(y')^2} \qquad y' = y - \frac{x}{N}$$

$$= e^{ax^2/2N} \sqrt{\frac{2\pi}{Na}}$$

Thus the integral can be written

$$\int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi/Na}}\, e^{-\frac{N}{2}ay^2 + axy} = e^{ax^2/2N}$$

(c) The partition function is

$$Z = \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} e^{-\beta H_\Omega}$$

$$= \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} e^{\beta H \sum_i S_i + \frac{J}{2N}\left(\sum_i S_i\right)^2}$$

We can multiply the summand by the factor

$$1 = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi/N\beta J}}\, e^{-\frac{1}{2}\beta N J y^2}$$

$$= \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi/N\beta J}}\, e^{-\frac{1}{2}\beta N J\left(y - \frac{1}{N}\sum_i S_i\right)^2}$$

After interchanging the order of summation and integration, the partition function becomes

$$Z = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi/N\beta J}} \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} e^{\beta H \sum_i S_i + \cancel{\frac{J}{2N}\left(\sum_i S_i\right)^2}}$$

$$\times\, e^{-\frac{1}{2}\beta N J\left[y^2 - \frac{2}{N} y \sum_i S_i + \cancel{\frac{1}{N^2}\left(\sum_i S_i\right)^2}\right]}$$

$$= \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi/N\beta J}}\, e^{-\frac{1}{2}\beta N J y^2} \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} e^{\beta(H + Jy)\sum_i S_i}$$

The sums can be evaluated analytically

$$\sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} e^{\beta(H+Jy)(S_1 + \cdots + S_N)}$$

$$= \prod_{i=1}^{N} \left(\sum_{S_i=\pm1} e^{\beta(H+Jy)S_i} \right)$$

$$= \left(\sum_{S_1 = \pm 1} e^{\beta (H + Jy) S_1} \right)^N$$

$$= \left(e^{\beta(H+Jy)} + e^{-\beta(H+Jy)} \right)^N$$

$$= \left(2 \cosh [\beta (H + Jy)] \right)^N$$

Inserting this back into the partition function, it becomes

$$Z = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi / N\beta J}} e^{-\frac{1}{2}\beta N J y^2} \left(2 \cosh [\beta (H + Jy)] \right)^N$$

$$= \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi / N\beta J}} e^{-N\beta L(y)}$$

where $L(y) = \frac{1}{2} J y^2 - \frac{1}{\beta} \log \left(2 \cosh [\beta (H + Jy)] \right)$

This expression can become nonanalytic if the

(d) In the thermodynamic limit $N \to \infty$, the integral can be evaluated by the method of steepest descents. The leading contributions as $N \to \infty$ are given by the values of the integrand at the stationary points y_i of the function $L(y)$:

$$Z = \sum_i e^{-\beta N L(y_i)}$$

The stationary points y_i of $L(y)$ satisfy

$$0 = \frac{1}{2} J \cdot (2y_i) - \frac{1}{\beta} \frac{1}{2\cosh[\beta(H+Jy_i)]} 2\sinh[\beta(H+Jy_i)] \cdot \beta J$$

$$y_i = \tanh[\beta(H+Jy_i)]$$

If $L(y)$ has a single global minimum y_0, its probability will be

$$\frac{e^{-N\beta L(y_0)}}{\sum_i e^{-N\beta L(y_i)}} = \frac{1}{1 + \sum_{i \neq 0} e^{-N\beta[L(y_i) - L(y_0)]}}$$

which approaches 1 as $N \to \infty$.

The magnetization is

$$M = \lim_{N \to \infty} \frac{1}{\beta N} \frac{\partial}{\partial H} \log Z$$

$$= \lim_{N\to\infty} \frac{1}{\beta N} \frac{\partial}{\partial H}\left(-N\beta L(y_0)\right)$$

$$= -\frac{\partial L}{\partial H}(y_0)$$

$$= -\left(-\frac{1}{\beta}\right)\frac{1}{2\cosh[\beta(H+Jy_0)]}\, 2\sinh[\beta(H+Jy_0)]\cdot\beta$$

$$= \tanh[\beta(H+Jy_0)]$$

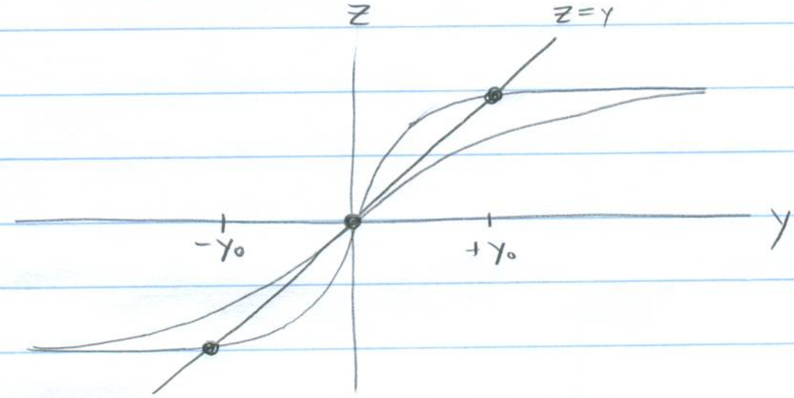
By the variational equation for y_0, this is simply

$$M = y_0$$

(e) If $H=0$, the variational equation reduces to

$$y = \tanh[\beta J y]$$

This can solved graphically by plotting $z=y$ and $z = \tanh[\beta J y]$ as functions of y and looking for intersections of the curves.



If $\beta J < 1$, the only simultaneous solution is $y = 0$.

If $\beta J > 1$, there are 3 simultaneous solutions: $y = 0$ and two nontrivial solutions, $y = \pm y_0$. The second derivative of $L(y)$ is

$$L''(y) = J - \frac{J}{\cosh^2[\beta J y]} \beta J$$

$$= J - \beta J^2 (1 - y^2)$$

In the last step, we used the variational equation, which implies

$$y^2 = \frac{\sinh^2(\beta J y)}{\cosh^2(\beta J y)} = 1 - \frac{1}{\cosh^2(\beta J y)}$$

At $y = 0$, the second derivative is negative:

$$L''(0) = J - \beta J^2$$

$$= J(1 - \beta J) < 0$$

Thus this is a local maximum of $L(y)$. The minima are $\pm y_0$.

The critical temperature T_c satisfies $\beta J = 1$

$$kT_c = J.$$

(f). The isothermal susceptibility is

$$\chi_T = \frac{\partial M}{\partial H}$$

The variational equation implies that the magnetization satisfies

$$M = \tanh[\beta(H + JM)]$$

Differentiating both sides, we get

$$\frac{\partial M}{\partial H} = \frac{1}{\cosh^2[\beta(H+JM)]}\left(\beta + \beta J \frac{\partial M}{\partial H}\right)$$

Solving for the derivative, we get

$$\frac{\partial M}{\partial H} = \frac{\beta}{\cosh^2[\beta(H+JM)]} \Big/ \left(1 - \frac{\beta J}{\cosh^2[\beta(H+JM)]}\right)$$

$$= \frac{\beta}{\cosh^2[\beta(H+JM)] - \beta J}$$

Thus the susceptibility at $H=0$ is

$$\chi_T = \frac{\beta}{\cosh^2(\beta J M) - \beta J}$$

For $T > T_c$, the magnetization is $M=0$, so the susceptibility is

$$\chi_T = \frac{\beta}{1-\beta J} = \frac{1}{k(T-T_c)}$$

This can be expressed as

$$\chi_T = \frac{1}{kT_c}\frac{1}{t}$$

where $t = \frac{T-T_c}{T_c}$. Thus χ_T diverges to ∞ as $T \to T_c^+$.

For $T < T_c$, the variational equation has a nonzero solution M. The variational equation can be written

$$M = \tanh[\beta J M]$$

$$= \tanh \frac{M}{1+t}$$

Expanding the right side in powers of M and t, we get

$$M = \frac{M}{1+t} - \frac{1}{3}\left(\frac{M}{1+t}\right)^3 + \cdots$$

$$= M(1-t+t^2+\cdots) - \frac{1}{3}M^3(1-3t+\cdots) + \cdots$$

Keeping terms suppressed by one power of t, this reduces to

$$0 = M(-t)(1-t) - \frac{1}{3}M^3(1-3t)$$

The nontrivial solution satisfies

$$M^2 = (-3t)(1+2t)$$

We can simplify the expression for the susceptibility by using

$$\cosh^2(\beta J M) = \frac{1}{1-M^2}$$

The susceptibility becomes

$$\chi_T = \frac{\beta}{\frac{1}{1-M^2} - \beta J}$$

$$= \frac{1}{kT_c} \frac{1-M^2}{1+t-(1-M^2)}$$

$$= \frac{1}{kT_c} \frac{1-M^2}{t+M^2}$$

As $T \to T_c^-$, the behavior of the susceptibility is

$$\chi_T = \frac{1}{kT_c} \frac{1-(-3t)}{t+(-3t)(1+2t)}$$

$$= \frac{1}{kT_c} \frac{1+3t}{-2t(1+3t)}$$

$$= \frac{1}{kT_c} \frac{1}{2} \frac{1}{|t|}$$

The corrections are of order t.