Goldenfeld, Chapter 6

Exercise 6-1

The Landau free energy is

$$L = \int d^d x \left\{ \tfrac{1}{2}\gamma \nabla\eta\cdot\nabla\eta + at\eta^2 + \tfrac{1}{2}b\eta^4 \right\} + a_0 V$$

For $T < T_c$, the minimum is at $\eta = \eta_s = \sqrt{\frac{a}{b}|t|}$

(a) The order parameter can be expanded around the minimum:

$$\eta(\vec{x}) = \eta_s + \psi(\vec{x})$$

The expansion of L to quadratic order in ψ is

$$L = \int d^d x \left\{ at\eta_s^2 + \tfrac{1}{2}b\eta_s^4 \right\} + a_0 V$$

$$+ \int d^d x \left\{ \tfrac{1}{2}\gamma \nabla\psi\cdot\nabla\psi + at\psi^2 + \tfrac{1}{2}b\cdot 6\eta_s^2\psi^2 \right\}$$

$$= \left(a_0 - \frac{a^2}{2b}t^2 \right) V$$

$$+ \int d^d x \left\{ \tfrac{1}{2}\gamma \nabla\psi\cdot\nabla\psi + 2a|t|\psi^2 \right\}$$

The linear terms in ψ cancel. To express this in terms of the Fourier components of ψ, we substitute

$$\psi(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \eta_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$$

The quadratic terms in L reduce to

$$L_2 = \frac{1}{V} \sum_{\vec{k}} \sum_{\vec{k}'} \left[\tfrac{1}{2} \gamma (i\vec{k})\cdot(i\vec{k}') + 2a|t| \right] \eta_{\vec{k}} \eta_{\vec{k}'} \underbrace{\int d^d x \, e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{V \delta_{\vec{k},\vec{k}'}}$$

$$= \sum_{\vec{k}} \left(\tfrac{1}{2} \gamma k^2 + 2a|t| \right) \eta_{\vec{k}} \eta_{-\vec{k}}$$

Thus the expansion of L reduces to

$$L = \left(a_0 - \frac{a^2}{2b} t^2\right) V + \sum_{\vec{k}} \tfrac{1}{2}\left(\gamma k^2 + 4a|t|\right) |\eta_{\vec{k}}|^2$$

(b) The partition function is

$$Z = \int \mathcal{D}\psi \, e^{-\beta L} = e^{-\beta F}$$

The Gaussian approximation to the functional integral can be calculated by repeating the steps from (6.42) to (6.47). The free energy is

$$F = \left(a_0 - \frac{a^2}{2b} t^2\right) - \tfrac{1}{2} k_B T \sum_{|\vec{k}|<\Lambda} \log \frac{2\pi V \cdot k_B T}{\gamma k^2 + 4a|t|}$$

In the continuum limit, the sum becomes an integral:

$$\sum_{|\vec{k}|<\Lambda} \longrightarrow V \int_{|\vec{k}|<\Lambda} \frac{d^d k}{(2\pi)^d}$$

The heat capacity per volume is

$$C = -\frac{T}{V}\frac{\partial^2 F}{\partial T^2}$$

The most singular contribution comes from both derivatives acting on the factor of $|t|$ in the denominator of the argument of the logarithm:

$$C_- = -\frac{T_c}{V}\left(-\tfrac{1}{2}k_B T_c\right) V \int_{|\mathbf{k}|<\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{(\gamma k^2 + 4a|t|)^2}\left(-\frac{4a}{T_c}\right)^2$$

$$= k_B \frac{8a^2}{\gamma^2}\frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\Lambda dk\, \frac{k^{d-1}}{(k^2+\xi^{-2})^2} \qquad \xi^2 = \frac{\gamma}{4a|t|}$$

The most singular contribution to the integral as $t \to 0$ is proportional to $(4a|t|)^{d/2-2}$ by dimensional analysis. It can be obtained by analytic continuation in d from the region $0<d<4$ where the integral converges for $\Lambda = \infty$:

$$C_- = k_B \frac{16}{(4\pi)^{d/2}\Gamma(d/2)}\frac{a^2}{\gamma^2}\int_0^\infty dk\, \frac{k^{d-1}}{(k^2+\xi^{-2})^2}$$

$$= k_B \frac{16}{(4\pi)^{d/2}\Gamma(d/2)}\frac{a^2}{\gamma^2}\cdot\tfrac{1}{2}\Gamma\left(\tfrac{d}{2}\right)\Gamma\left(2-\tfrac{d}{2}\right)\xi^{4-d}$$

$$= k_B \frac{8\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\frac{a^2}{\gamma^2}\left(\frac{\gamma}{4a|t|}\right)^{2-d/2}$$

This depends on the parameters a and γ in the Landau free energy.

The most singular contribution c_+ for $T > T_c$ can be obtained by differentiating the expression for the free energy in Eq. (6.47):

$$F = a_0 V - \tfrac{1}{2} k_B T \sum_{\vec{k}} \log \frac{2\pi V \cdot k_B T}{\gamma k^2 + 2at}$$

c_+ differs from c_- by a factor of $(\tfrac{1}{2})^2$ from the two time derivatives and by the expression for the correlation length

$$C_+ = k_B \frac{4}{(4\pi)^{d/2}\Gamma(d/2)} \frac{a^2}{\gamma^2} \int_0^\infty dk \frac{k^{d-1}}{(k^2+\xi^{-2})^2} \qquad \xi^2 = \frac{\gamma}{2at}$$

$$= k_B \frac{2\Gamma(2-d/2)}{(4\pi)^{d/2}} \frac{a^2}{\gamma^2} \left(\frac{\gamma}{2at}\right)^{2-d/2}$$

The ratio c_+ and c_- for the same values of $|t|$ is

$$\frac{C_+}{C_-} = \frac{1}{4} 2^{2-d/2}$$

$$= 2^{-d/2}$$

This ratio is independent of the parameters in the Landau free energy.

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Exercise 6-2

A plausible criterion is

$$\langle (\eta-\eta_s)^2 \rangle \ll \langle \eta^2 \rangle \approx \eta_s^2$$

The left side can be written as

$$\frac{1}{V}\int d^d r \left\langle (\eta(\vec{r})-\eta_s)^2 \right\rangle = \frac{1}{V}\int d^d r \langle \psi(\vec{r})^2 \rangle$$

$$= \frac{1}{V}\int d^d r \left\langle \left(\frac{1}{\sqrt{V}}\sum_{\vec{k}} \eta_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\right)\left(\frac{1}{\sqrt{V}}\sum_{\vec{k}'} \eta_{\vec{k}'} e^{i\vec{k}'\cdot\vec{r}}\right)\right\rangle$$

$$= \frac{1}{V^2}\sum_{\vec{k}}\sum_{\vec{k}'} \langle \eta_{\vec{k}}\eta_{\vec{k}'}\rangle \underbrace{\int d^d\vec{r}\, e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}}_{V\delta_{\vec{k}+\vec{k}',0}}$$

$$= \frac{1}{V}\sum_{\vec{k}} \langle \eta_{\vec{k}}\eta_{-\vec{k}}\rangle$$

$$= \frac{1}{V}\sum_{\vec{k}} \langle |\eta_{\vec{k}}|^2\rangle$$

The expectation value of $|\eta_{\vec{k}}|^2$ is given in Eq. (6.61). Thus the left side can be written

$$\langle (\eta-\eta_s)^2\rangle = \frac{1}{V}\sum_{\vec{k}} \frac{k_B T V}{2at+\gamma \vec{k}^2}$$

In the thermodynamic limit, the sum can be replaced by an integral:

$$\langle(\eta-\eta_s)^2\rangle = k_B T \int_{|k|<\Lambda} \frac{d^dk}{(2\pi)^d}\,\frac{1}{2at+\gamma k^2}$$

$$= \frac{k_B T}{\gamma}\,\frac{1}{(2\pi)^d}\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\Lambda dk\,k^{d-1}\,\frac{1}{k^2+\xi^{-2}}$$

where $\xi^{-2} = \frac{2at}{k}$. The leading contribution to the integral in the limit $t \to 0$ is given by the analytic continuation in d of the integral with $\Lambda = \infty$:

$$\int_0^\infty dk\,k^{d-1}\,\frac{1}{k^2+\xi^{-2}} = \frac{\pi}{2\sin(\pi d/2)}\,\xi^{2-d}$$

$$= \frac{1}{2}\,\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\xi^{2-d}$$

Thus the expectation value reduces to

$$\langle(\eta-\eta_s)^2\rangle = \frac{k_B T}{\gamma}\,\frac{2}{(4\pi)^{d/2}\,\Gamma(d/2)}\,\frac{1}{2}\,\Gamma(d/2)\,\Gamma\left(1-\frac{d}{2}\right)\xi^{2-d}$$

$$= \frac{k_B T}{\gamma}\,\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\,\xi^{2-d}$$

Using $\eta_s^2 = \frac{a}{b}|t|$ and $\xi^2 = \frac{\gamma}{4a|t|}$, our criterion becomes

$$\frac{k_B T_c}{\gamma}\,\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\left(\frac{\gamma}{4a|t|}\right)^{1-d/2} \ll \frac{a}{b}|t|$$

$$|t|^{2-d/2} \gg \frac{\Gamma(1-d/2)}{4\,(4\pi)^{d/2}}\left(\frac{4a}{\gamma}\right)^{d/2}\frac{b}{a^2}\,kT_c$$

This is the same as the Ginsburg criterion up to numerical prefactors.

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Exercise 6-3

The partition function is

$$Z = \int \mathcal{D}\psi \, e^{-\beta S[\psi]}$$

where S is a functional of $\psi_i(\vec{r})$, $i = 1, 2, \ldots, N$, that depends on the parameters H_i and J_{ij}:

$$S[\psi] = \tfrac{1}{2}(\psi_i - H_i) J_{ij}^{-1} (\psi_j - H_j) - \frac{1}{\beta} \sum_i \log\left[2\cosh(\beta\psi_i)\right]$$

(a) We expand $S[\psi]$ to 2^{nd} order in $\eta_i = \psi_i - \Psi_i$, where Ψ is the stationary point of S, which satisfies

$$J_{ij}^{-1}(\Psi_j - H_j) = \tanh(\beta\Psi_i)$$

The expansion is

$$S[\psi] = S[\Psi] + \tfrac{1}{2}\eta_i J_{ij}^{-1}\eta_j - \tfrac{1}{2}\beta \sum_i \left[1 - \tanh^2(\beta\Psi_i)\right]\eta_i^2$$

$$= S[\Psi] + \tfrac{1}{2}\sum_{ij} \eta_i \left(J_{ij}^{-1} - \beta\left[1 - \tanh^2(\beta\Psi_i)\right]\delta_{ij}\right)\eta_j$$

The partition function is

$$Z = e^{-\beta S[\bar\Psi]} \int \mathcal{D}\eta \, \exp\left(-\frac{1}{\beta}\sum_i \eta_i \left(J_{ij}^{-1} - \beta[1-\tanh^2(\beta\bar\Psi_i)]\right)\eta_j\right)$$

$$= e^{-\beta S[\bar\Psi]} \left[\det\left(\frac{1}{\beta}J_{ij}^{-1} - [1-\tanh^2(\beta\bar\Psi_i)]\delta_{ij}\right)\right]^{-1/2}$$

$$\times \int \mathcal{D}\eta \, \exp\left(-\sum_i \eta_i^2\right)$$

$$= e^{-\beta S[\bar\Psi]} \left[\det\left(\delta_{ij} - [1-\tanh^2(\beta\bar\Psi_i)]J_{ij}\right)\right]^{-1/2}$$

$$\times \left[\det\left(\frac{1}{\beta}J_{ij}^{-1}\right)\right]^{-1/2} \int \mathcal{D}\eta \, \exp\left(-\sum_i \eta_i^2\right)$$

The last two factors are an irrelevant multiplicative constant in the sense that they do not depend on H_i and they depend smoothly on T near T_c. The free energy is therefore

$$F[H] = S[\bar\Psi] + \frac{1}{2\beta}\log\left[\det\left(\delta_{ij} - [1-\tanh^2(\beta\bar\Psi_i)]J_{ij}\right)\right]$$

(b) The Gibbs free energy is

$$\Gamma[m] = F[H(m)] + \sum_i H_i(m)\, m_i$$

where H must be eliminated in favor of the magnetization m_i, $i=1,\ldots,N$, which satisfies

$$\frac{\partial F}{\partial H_i} = -m_i$$

This condition implies that the right side of the expression for $\Gamma[m]$ is a stationary function of H.

We express the free energy as

$$F[H] = S[\bar{\Psi}] + \epsilon \, \delta F[H]$$

where $\epsilon = 1$ keeps track of the order of the approximation. The derivative with respect to H_i is

$$\frac{\partial F}{\partial H_i} = - J_{ij}^{-1} (\bar{\Psi}_j - H_j) + \epsilon \frac{\partial \delta F}{\partial H_i}$$

where we have used the fact that $\bar{\Psi}$ is a stationary point of $S[\Psi]$. Thus the magnetization is

$$m_i = J_{ij}^{-1} (\bar{\Psi}_j - H_j) - \epsilon \frac{\partial \delta F}{\partial H_i}$$

We can solve this for H:

$$H_i = \bar{\Psi}_i - J_{ij}^{-1} \left(m_j - \epsilon \frac{\partial \delta F}{\partial H_i} \right)$$

Since H is a stationary point of the expression for $\Gamma[m]$, the term of order ϵ in H only contributes to $\Gamma[m]$ at order ϵ^2. The Gaussian approximation to $\Gamma[m]$ is the expansion to order. To this accuracy, we can set $\epsilon = 0$ in the expression for H:

$$H_i = \Phi_i - J_{ij}^{-1} m_j$$

Using the stationary condition for Φ_i, this implies

$$\tanh(\beta \Phi_i) = m_i$$

The mean-field term in the Gibbs free energy was determined in Problem 3-3:

$$S[\Phi] + H_i m_i \Big|_{H_i = \Phi_i - J_{ij}^{-1} m_j}$$

$$= -\frac{1}{2} m_i J_{ij} m_j + \frac{1}{\beta} \sum_i \left(\frac{1+m_i}{2} \log \frac{1+m_i}{2} + \frac{1-m_i}{2} \log \frac{1-m_i}{2} \right)$$

We can set $\tanh(\beta\Phi_i) = m_i$ in the Gaussian correction term. Thus the Gibbs free energy in the Gaussian approximation is

$$\Gamma[m] = -\frac{1}{2} m_i J_{ij} m_j + \frac{1}{\beta} \sum_i \left(\frac{1+m_i}{2} \log \frac{1+m_i}{2} + \frac{1-m_i}{2} \log \frac{1-m_i}{2} \right)$$

$$+ \frac{1}{2\beta} \log\left[\det\left(\delta_{ij} - (1-m_i^2) J_{ij} \right) \right]$$

(c) I did not succeed in solving the part.