Trapped Bosonic Atoms

N bosonic atoms
all in the same spin state
fixed total energy U
trapped in harmonic oscillator potential

$$V(x,y,z) = \frac{1}{2} m\omega^2 r^2 = \frac{1}{2} m\omega^2 (x^2+y^2+z^2)$$

thermodynamic limit: $N \to \infty$, $U \to \infty$, $\frac{U}{N}$ fixed

use Grand Canonical Ensemble
with temperature T
chemical potential μ

<u>lowest energy orbital</u>

energy: $\epsilon_0 = \frac{3}{2}\hbar\omega$

wavefunction: $\psi_0(\vec{r}) = \left(\frac{1}{\pi a^2}\right)^{3/4} e^{-r^2/2a^2}$ $\quad a = \sqrt{\frac{\hbar}{m\omega}}$

Fourier transform: $\tilde{\psi}_0(\vec{p}) = \left(\frac{a^2}{\pi}\right)^{3/4} e^{-a^2 p^2/2}$

$$\langle x^2 \rangle_0^{1/2} = \frac{1}{\sqrt{2}} a$$

$$\langle p_x^2 \rangle_0^{1/2} = \frac{1}{\sqrt{2}} \frac{\hbar}{a}$$

<u>other orbitals</u>

label by $(\vec{r}, \vec{p}) = (x, y, z, p_x, p_y, p_z)$

energy: $\epsilon(r,p) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 r^2$
$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}m\omega^2(x^2+y^2+z^2)$

sum over orbitals: $\frac{1}{h^3}\int d^3r\, d^3p$

average occupation numbers: $n_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$

<u>$T > T_c$</u>

$$\boxed{\begin{aligned} N &= \frac{1}{h^3}\int d^3r\, d^3p\, \frac{1}{e^{\beta[\epsilon(r,p)-\mu]}-1} \\ U &= \frac{1}{h^3}\int d^3r\, d^3p\, \frac{\epsilon(r,p)}{e^{\beta[\epsilon(r,p)-\mu]}-1} \end{aligned}}$$

<u>$T < T_c$</u> trade μ for $N_0 = \frac{1}{e^{\beta(\epsilon_0-\mu)}-1}$

$$\boxed{\begin{aligned} N &= N_0 + \frac{1}{h^3}\int d^3r \int d^3p\, \frac{1}{e^{\beta\epsilon(r,p)}-1} \\ U &= N_0\epsilon_0 + \frac{1}{h^3}\int d^3r \int d^3p\, \frac{1}{e^{\beta\epsilon(r,p)}-1} \end{aligned}}$$

express sum over orbitals
as integral over orbital energy

$$\frac{1}{h^3}\int d^3r\, d^3p = \frac{1}{h^3}\int dx\, dy\, dz \int dp_x\, dp_y\, dp_z$$

$$\epsilon = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}m\omega^2\left(x^2+y^2+z^2\right)$$

define
$$\begin{aligned} P_1 &= p_x & P_4 &= m\omega x \\ P_2 &= p_y & P_5 &= m\omega y \\ P_3 &= p_z & P_6 &= m\omega z \end{aligned}$$

$$\frac{1}{h^3}\int d^3r\, d^3p = \frac{1}{h^3}\frac{1}{(m\omega)^3}\int dP_1\, dP_2\, dP_3\, dP_4\, dP_5\, dP_6$$

$$= \frac{1}{(m\omega h)^3}\int d^6P$$

$$= \frac{1}{(m\omega h)^3}\int P^5\, dP\, d\Omega_6$$

angular integral: $\displaystyle\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$

$$\int d\Omega_6 = \pi^3$$

$$\epsilon = \frac{1}{2m}\left(P_1^2 + P_2^2 + P_3^2 + P_4^2 + P_5^2 + P_6^2\right) = \frac{1}{2m}P^2$$

momentum integral: $\displaystyle\int P^5\, dP = \frac{1}{2}(2m)^3\int \epsilon^2\, d\epsilon$

$$\boxed{\frac{1}{h^3}\int d^3r\, d^3p = \frac{1}{2}\frac{1}{(\hbar\omega)^3}\int_0^\infty d\epsilon\, \epsilon^2}$$

integrals for $T < T_c$

$$\frac{1}{h^3}\int d^3r\, d^3p\, \frac{1}{e^{\beta\epsilon}-1} = \frac{1}{2}\frac{1}{(\hbar\omega)^3}\int_0^\infty d\epsilon\, \epsilon^2 \frac{1}{e^{\beta\epsilon}-1}$$

$$= \frac{1}{2}\frac{1}{(\hbar\omega)^3}\frac{1}{\beta^3}\underbrace{\int_0^\infty dx\, \frac{x^2}{e^x-1}}_{=2\zeta(3)} \qquad \zeta(3)\approx 1.20$$

$$= \zeta(3)\left(\frac{kT}{\hbar\omega}\right)^3$$

$$\frac{1}{h^3}\int d^3r\, d^3p\, \frac{\epsilon}{e^{\beta\epsilon}-1} = \frac{1}{2}\frac{1}{(\hbar\omega)^3}\int_0^\infty d\epsilon\, \epsilon^2 \frac{\epsilon}{e^{\beta\epsilon}-1}$$

$$= \frac{1}{2}\frac{1}{(\hbar\omega)^3}\frac{1}{\beta^4}\underbrace{\int_0^\infty dx\, \frac{x^3}{e^x-1}}_{=\frac{\pi^4}{15}}$$

$$= \frac{\pi^4}{30}\frac{(kT)^4}{(\hbar\omega)^3}$$

critical temperature

$$N = \frac{1}{h^3}\int d^3p\, d^3r\, \frac{1}{e^{\epsilon/kT_c}-1}$$

$$= \zeta(3)\left(\frac{kT_c}{\hbar\omega}\right)^3$$

$$\boxed{kT_c = \left(\frac{N}{\zeta(3)}\right)^{1/3}\hbar\omega}$$

condensate fraction for $T < T_c$

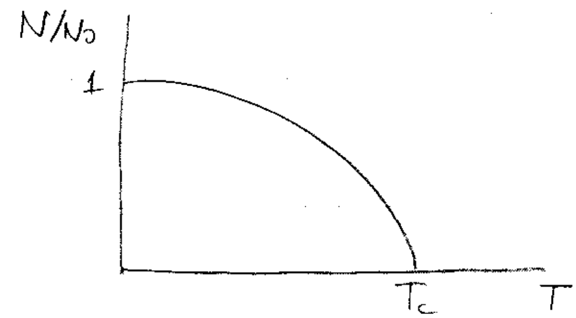
$$N = N_0 + \frac{1}{h^3}\int d^3r\, d^3p \, \frac{1}{e^{\beta\epsilon}-1}$$

$$= N_0 + \zeta(2)\left(\frac{kT}{\hbar\omega}\right)^3$$

$$= N_0 + N\left(\frac{T}{T_c}\right)^3$$

$$N_0 = N - N\left(\frac{T}{T_c}\right)^3$$

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3}$$



energy for $T < T_c$

$$U = N_0\epsilon_0 + \frac{1}{h^3}\int d^3r\, d^3p \, \frac{\epsilon}{e^{\beta\epsilon}-1}$$

$$= N_0\left(\frac{3}{2}\hbar\omega\right) + \frac{\pi^4}{30}\frac{(kT)^4}{(\hbar\omega)^3}$$

$$= \left(\frac{3}{2}\hbar\omega\right)\cdot N\left[1-\left(\frac{T}{T_c}\right)^3\right] + \frac{\pi^4}{30\zeta(3)}kT\cdot N\left(\frac{T}{T_c}\right)^3$$

condensed atoms have energy $\frac{3}{2}\hbar\omega$

noncondensed atoms have average energy $\frac{\pi^4}{30\zeta(3)}kT \approx 2.7\, kT$

distribution in p_x, p_y

high temperature $T \gg T_c$

canonical ensemble $\Rightarrow$ Gaussian distribution

equipartition theorem: $\left\langle \frac{p_x^2}{2m} \right\rangle = \left\langle \frac{p_y^2}{2m} \right\rangle = \frac{1}{2} kT$

$$\langle p_x^2 \rangle^{\frac{1}{2}} = \langle p_y^2 \rangle^{\frac{1}{2}} = \sqrt{mkT}$$

below T_c

condensate atoms: Gaussian wavefunction

$$\tilde{\psi} \propto e^{-a^2(p_x^2+p_y^2)/2} \qquad a = \sqrt{\frac{\hbar}{m\omega}}$$

$$\langle p_x^2 \rangle^{1/2} = \langle p_y^2 \rangle^{1/2} = \frac{1}{\sqrt{2}} \sqrt{\hbar m \omega}$$

non-condensate atoms: Bose-Einstein distribution with 0 chemical potential

$$\left\langle \frac{p_x^2}{2m} \right\rangle = \left\langle \frac{p_y^2}{2m} \right\rangle = \frac{1}{6} \langle \epsilon \rangle = \frac{1}{6} \cdot \frac{\pi^4}{30\zeta(3)} kT$$

$$\langle p_x^2 \rangle^{1/2} = \langle p_y^2 \rangle^{1/2} = \sqrt{\frac{\pi^4}{90\zeta(3)} mkT} \approx 0.95 \sqrt{mkT}$$