Trapped Fermionic Atoms

N identical fermionic atoms
all in the same spin state

trapped in harmonic oscillator potential

$$V(x,y,z) = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2(x^2+y^2+z^2)$$

thermodynamic limit: $N \to \infty, U \to \infty, \frac{U}{N}$ fixed

use Grand Canonical Ensemble
with temperature T
chemical potential μ

orbital

label by $(\vec{r},\vec{p}) = (x,y,z,p_x,p_y,p_z)$

energy: $\epsilon = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$

$$= \frac{1}{2m}(p_x^2+p_y^2+p_z^2) + \frac{1}{2}m\omega^2(x^2+y^2+z^2)$$

sum over orbital: $\frac{1}{h^3}\int d^3r\, d^3p$

average occupation numbers: $n_{FD}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$

express sum over orbitals
as integral over orbital energy

$$\frac{1}{h^3}\int d^3r\, d^3p = \frac{1}{h^3}\int dx\, dy\, dz \int dp_x\, dp_y\, dp_z$$

$$\epsilon = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}m\omega^2\left(x^2+y^2+z^2\right)$$

define
$$P_1 = p_x \qquad P_4 = m\omega x$$
$$P_2 = p_y \qquad P_5 = m\omega y$$
$$P_3 = p_z \qquad P_6 = m\omega z$$

$$\frac{1}{h^3}\int d^3r\, d^3p = \frac{1}{h^3}\frac{1}{(m\omega)^3}\int dP_1\, dP_2\, dP_3\, dP_4\, dP_5\, dP_6$$

$$= \frac{1}{(m\omega h)^3}\int d^6P$$

$$= \frac{1}{(m\omega h)^3}\int P^5\, dP\, d\Omega_6$$

angular integral: $\int d\Omega_d = \dfrac{2\pi^{d/2}}{\Gamma(d/2)}$

$$\int d\Omega_6 = \pi^3$$

$$\epsilon = \frac{1}{2m}\left(P_1^2 + P_2^2 + P_3^2 + P_4^2 + P_5^2 + P_6^2\right) = \frac{1}{2m}P^2$$

momentum integral: $\int P^5\, dP = \frac{1}{2}(2m)^3 \int \epsilon^2\, d\epsilon$

$$\boxed{\frac{1}{h^3}\int d^3r\, d^3p = \frac{1}{2}\frac{1}{(\hbar\omega)^3}\int_0^\infty d\epsilon\, \epsilon^2}$$

number of atoms

$$\boxed{N = \frac{1}{h^3}\int d^3r\, d^3p\, \frac{1}{e^{\beta(\epsilon-\mu)}+1}}$$

solve for μ as a function of N

total energy

$$\boxed{U = \frac{1}{h^3}\int d^3r\, d^3p\, \frac{\epsilon}{e^{\beta(\epsilon-\mu)}+1}}$$

eliminate μ

Low-temperature limit

all orbitals filled up to the Fermi energy ϵ_F
all higher orbitals unoccupied

$$n_{FD}(\epsilon) \longrightarrow 1 \qquad \epsilon < \epsilon_F$$
$$\longrightarrow 0 \qquad \epsilon > \epsilon_F$$

$$N = \frac{1}{2}\frac{1}{(\hbar\omega)^3}\int_0^{\epsilon_F} d\epsilon\, \epsilon^2$$

$$= \frac{1}{2}\frac{1}{(\hbar\omega)^3}\frac{\epsilon_F^3}{3}$$

$$= \frac{1}{6}\left(\frac{\epsilon_F}{\hbar\omega}\right)^3$$

Fermi energy: $\boxed{\epsilon_F = (6N)^{1/3}\hbar\omega}$

total energy

$$U = \frac{1}{2(\hbar\omega)^3} \int_0^{\epsilon_F} d\epsilon\, \epsilon^2 \cdot \epsilon$$

$$= \frac{1}{2(\hbar\omega)^2} \frac{\epsilon_F^4}{4}$$

$$= N\left(\tfrac{3}{4}\epsilon_F\right)$$

average energy: $\langle \epsilon \rangle = \frac{3}{4}\epsilon_F$

atoms have maximum radius

$$\frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2 < \epsilon_F$$

$$\Longrightarrow r < \sqrt{\frac{2\epsilon_F}{m\omega^2}} \equiv R_F$$

"Fermi radius": $R_F = (6N)^{1/6} \sqrt{\frac{2\hbar}{m\omega}}$

atoms have maximum momentum

$$\frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2 < \epsilon_F$$

$$\Longrightarrow p < \sqrt{2m\epsilon_F} \equiv p_F$$

"Fermi momentum": $p_F = (6N)^{1/6} \sqrt{2m\hbar\omega}$