Baierlein, Chapter 9

Problem 7

(a) The ideal gas of N fermions in volume V_i at zero temperature have total energy

$$U = \frac{3}{5} N \epsilon_F, \quad \text{where } \epsilon_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V_i}\right)^{2/3}$$

If it expands adiabatically to a final volume V_f, the total energy will be unchanged. If it is a classical ideal gas, the total energy is

$$U = \frac{3}{2} N k T_f$$

Thus the final temperature is

$$kT_f = \frac{2}{5}\epsilon_F$$

(b) If the final state is a classical ideal gas, the cube of the thermal de Broglie wavelength must be much smaller than the volume per particle

$$\left(\frac{h}{\sqrt{2\pi m k T_f}}\right)^3 \ll \frac{V_f}{N}$$

On the left side, kT_f can be eliminated in favor of V_i/N:

$$\frac{h^3}{(2\pi m k T_f)^{3/2}} = \frac{h^3}{\left[2\pi m \frac{2}{5}\cdot\frac{h^2}{2m}\left(\frac{3N}{8\pi V_i}\right)^{2/3}\right]^{3/2}}$$

$$= \left(\frac{5}{2\pi}\right)^{3/2} \frac{8\pi V_i}{3N}$$

The inequality can therefore be written

$$\frac{V_f}{V_i} >> \left(\frac{5}{2\pi}\right)^{3/2} \frac{8\pi}{3}$$

(c) If "heating" refers to the flow of thermal energy, the title is a contradiction, because "adiabatic" implies that there is no flow of thermal energy in or out of the gas.

If "heating" is used in the sense of increase of temperature, the title may be legitimate.

Baierlein, Chapter 9

Problem 19

(a) The minimum kinetic energy T in the center-of-mass frame required for the reaction $e^- + p \rightarrow n + \nu_e$ correspond to producing a neutron at rest and a neutrino with 0 energy. Conservation implies that T satisfies

$$m_e c^2 + m_p c^2 + T_{CM} = m_n c^2 + 0$$

The minimum kinetic energy is

$$T_{CM} = (m_n - m_p - m_e) c^2$$

$$= (939.56 - 938.27 - 0.511) \text{ MeV}$$

$$= 0.782 \text{ MeV}$$

b) The minimum kinetic energy T is the rest frame of the proton is approximately the same as in the center-of-momentum frame:

$$T = 0.782 \text{ MeV}$$

In this case, T is the kinetic energy of the electron.

Its total energy is the sum of T and its rest energy $m_e c^2$:

$$E = m_e c^2 + T$$

$$= (0.782 + 0.511)\,\mathrm{MeV}$$

$$= 1.293\ \mathrm{MeV}$$

The corresponding momentum p_F satisfies

$$E = \sqrt{(m_e c^2)^2 + (p_F c)^2}$$

$$p_F c = \sqrt{E^2 - (m_e c^2)^2}$$

$$= \sqrt{(1.293\ \mathrm{MeV})^2 - (0.511\ \mathrm{MeV})^2}$$

$$= 1.188\ \mathrm{MeV}$$

The number of electrons that correspond to the Fermi momentum p_F is

$$N_e = 2\frac{V}{h^3}\int_0^{p_F} 4\pi p^2\,dp$$

$$= \frac{2V}{(2\pi\hbar)^3}\,4\pi\,\frac{p_F^3}{3}$$

$$= \frac{1}{3\pi^2}\,V\,\frac{(p_F c)^3}{(\hbar c)^3}$$

The number density of electron is

$$n_e = \frac{N_e}{V}$$

$$= \frac{1}{3\pi^2}\left(\frac{p_F c}{\hbar c}\right)^3$$

$$= \frac{1}{3\pi^2}\left(\frac{1.188 \text{ MeV}}{197.3 \text{ MeV}\cdot\text{fm}}\right)^3$$

$$= 7.37 \cdot 10^{-9}/\text{fm}^3$$

$$= 7.37 \cdot 10^{36}/\text{m}^3$$

(c) The specified condition $N_e = N_p + N_n$ is unphysical unless $N_n = 0$, because electric charge neutrality requires $N_e = N_p$. A more appropriate condition would be $N_e = N_p = N_n$.

In general, the mass density is

$$\rho = m_e \frac{N_e}{V} + m_p \frac{N_p}{V} + m_n \frac{N_n}{V}$$

$$\approx 2 m_p \frac{N_e}{V}$$

Inserting the minimum electron density for which the reaction $e^- + p \to n_e + \nu_e$ is possible, we get

$$\rho = 2\,(1.67 \times 10^{-27}\ \text{kg})\,(7.37 \times 10^{36}/\text{m}^3)$$

$$= 2.46 \times 10^{10}\ \text{kg}/\text{m}^3$$

This is about 7 orders of magnitude larger than the average mass density of the earth, which is $5.5 \times 10^3\ \text{kg}/\text{m}^3$.

According to Wikipedia, the mass density in the crust of a neutron star is about $10^9\ \text{kg}/\text{m}^3$. Our estimate of the minimum mass density is about an order of magnitude higher.