**Physics 622 2nd Midterm Exam** NAME ________________________________

**Problem 1.**

The dark matter of the universe could be a Bose-Einstein condensate of *axions*. These hypothetical elementary particles are bosons and they are essentially massless: $m = 0$. Let $T_c$ be the critical temperature for Bose-Einstein condensation of the axions.

(A) Write down the occupation number of an orbital as a function of the energy $\epsilon$ of an axion. What is the energy of an orbital with momentum $\boldsymbol{p}$ and position $\boldsymbol{r}$?

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

$$\epsilon = \sqrt{p_x^2+p_y^2+p_z^2}\,c = pc$$

(B) For $T > T_c$, write down expressions for the number $N$ of axions and their total energy $U$ in terms of integrals that depend on the temperature $T$ and the chemical potential $\mu$.

$$N = \frac{V}{h^3}\int d^3p \, \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

$$U = \frac{V}{h^3}\int d^3p \, \frac{\epsilon}{e^{\beta(\epsilon-\mu)}-1}$$

(C) For $T < T_c$, write down expressions for $N$ and $U$ in terms of the number $N_0$ of axions in the Bose-Einstein condensate and an integral that depends on $T$.

$$N = N_0 + \frac{V}{h^3}\int d^3p \, \frac{1}{e^{\beta\epsilon}-1}$$

$$U = N_0 \cdot 0 + \frac{V}{h^3}\int d^3p \, \frac{\epsilon}{e^{\beta\epsilon}-1}$$

(D) Deduce the critical temperature $T_c$ as a function of $N/V$.

$$N = \frac{V}{h^3}\int d^3p \, \frac{1}{e^{\beta\epsilon}-1} = \frac{V}{h^3} 4\pi \int_0^\infty p^2 dp \, \frac{1}{e^{\beta\epsilon}-1}$$

$$= \frac{4\pi V}{h^3}\frac{1}{c^3}\int_0^\infty \epsilon^2 d\epsilon \, \frac{1}{e^{\beta\epsilon}-1} = \frac{4\pi V}{h^3}\frac{1}{(\beta c)^3}\int_0^\infty dt \, \frac{t^2}{e^t-1}$$

$$kT_c = \left[\frac{h^3c^3N}{4\pi V}\Big/\int_0^\infty dt \, \frac{t^2}{e^t-1}\right]^{1/3} = \left[4\pi\int_0^\infty dt \, \frac{t^2}{e^t-1}\right]^{-1/3} hc\left(\frac{N}{V}\right)^{1/3}$$

(E) For $T < T_c$, deduce the condensate number $N_0$ as a function of $T$.

$$N = N_0 + \left(4\pi\int_0^\infty dt \, \frac{t^2}{e^t-1}\right) V\left(\frac{kT}{hc}\right)^3$$

$$= N_0 + N\left(\frac{T}{T_c}\right)^3$$

$$N_0 = \left[1-\left(\frac{T}{T_c}\right)^3\right]N$$

**Problem 2.**

Consider a single orbital of energy $\epsilon$ in equilibrium with a reservoir of temperature $T$ and chemical potential $\mu$.

Suppose the orbital is the energy level of a boson.

(A) What are the possible occupation numbers $n$ of the orbital?

$$n = 0, 1, 2, \ldots$$

(B) What is the average occupation number $\bar{n}$ of the orbital?

$$\bar{n} = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}$$

(C) Express the grand partition function $\mathcal{Z}$ for the orbital as a sum over the occupation number and evaluate the sum.

$$\mathcal{Z} = \sum_{n=0}^{\infty} e^{-\beta(n\epsilon) + \beta\mu \cdot n}$$

$$= \sum_{n=0}^{\infty} \left(e^{-\beta(\epsilon-\mu)}\right)^n = \frac{1}{1 - e^{-\beta(\epsilon-\mu)}}$$

(D) Express the average occupation number $\bar{n}$ in terms of a derivative of $\mathcal{Z}$.

$$\bar{n} = \frac{1}{\beta}\left(\frac{\partial}{\partial\mu} \log \mathcal{Z}\right)_\beta$$

Suppose the orbital is the energy level of a fermion.

(E) What are the possible occupation numbers $n$ of the orbital?

$$n = 0, 1$$

(F) What is the average occupation number $\bar{n}$ of the orbital?

$$\bar{n} = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}$$

(G) Express the grand partition function $\mathcal{Z}$ for the orbital as a sum over the occupation number and evaluate the sum.

$$\mathcal{Z} = \sum_{n=0}^{1} e^{-\beta(n\epsilon) + \beta\mu n}$$

$$= \sum_{n=0}^{1} \left(e^{-\beta(\epsilon-\mu)}\right)^n = 1 + e^{-\beta(\epsilon-\mu)}$$

(H) Express the average occupation number $\bar{n}$ in terms of a derivative of $\mathcal{Z}$.

$$\bar{n} = \frac{1}{\beta}\left(\frac{\partial}{\partial\mu} \log \mathcal{Z}\right)_\beta$$

**Problem 3.**

A large number $N$ of <u>fermionic</u> atoms, all in the same spin state, are trapped in a 3-dimensional harmonic oscillator potential: $V(r) = \frac{1}{2}m\omega^2 r^2$.

(A) Write down the occupation number of an orbital as a function of the energy $\epsilon$ of an atom. What is the energy for an orbital with momentum $\boldsymbol{p}$ and position $\boldsymbol{r}$?

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

$$\epsilon = \frac{1}{2m}(p_x^2+p_y^2+p_z^2) + \frac{1}{2}m\omega^2(x^2+y^2+z^2) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 r^2$$

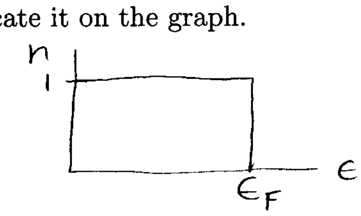
(B) Write down expressions for $N$ and the total energy $U$ of this system in terms of integrals that depend on the temperature $T$ and the chemical potential $\mu$.

$$N = \int \frac{d^3r\, d^3p}{h^3} \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

$$U = \int \frac{d^3r\, d^3p}{h^3} \frac{\epsilon}{e^{\beta(\epsilon-\mu)}+1}$$

For parts (C-E), consider the system at <u>zero temperature</u>.

(C) Draw a graph of the occupation number as a function of $\epsilon$. Define the Fermi energy $\epsilon_F$ and indicate it on the graph.




The Fermi energy $\epsilon_F$ is the largest energy for all the occupied orbitals.

(D) What is the largest momentum $p$ for any of the atoms? What is the largest radial coordinate $r$ for any of the atoms? (Express them in terms of $\epsilon_F$.)

$$\frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 r^2 < \epsilon_F \implies \frac{1}{2m}p^2 < \epsilon_F \implies p < \sqrt{2m\epsilon_F}$$

$$\implies \frac{1}{2}m\omega^2 r^2 < \epsilon_F \implies r < \sqrt{\frac{2\epsilon_F}{m\omega^2}}$$

(E) Determine the Fermi energy $\epsilon_F$ as a function of $N$ and $\omega$.

$$N = \int \frac{d^3r\, d^3p}{h^3}\Bigg|_{\epsilon<\epsilon_F} \qquad \begin{array}{ll} P_1 = P_x & P_4 = m\omega x \\ P_2 = P_y & P_5 = m\omega y \\ P_3 = P_z & P_6 = m\omega z \end{array}$$

$$= \frac{1}{h^3}\frac{1}{(m\omega)^3}\int dP_1\, dP_2\, dP_3\, dP_4\, dP_5\, dP_6 \Bigg|_{P_1^2+P_2^2+P_3^2+P_4^2+P_5^2+P_6^2<2m\epsilon_F}$$

$$= \frac{1}{(mh\omega)^3}\int d^6P \Bigg|_{P^2<2m\epsilon_F}$$

$$= \frac{1}{(mh\omega)^3}\pi^3 \int_0^{\sqrt{2m\epsilon_F}} P^5\, dP$$

$$= \left(\frac{\pi}{mh\omega}\right)^3 \frac{1}{6}(2m\epsilon_F)^3 = \frac{1}{6}\left(\frac{2\pi}{h\omega}\right)^3 \epsilon_F^3 \qquad \epsilon_F = (6N)^{1/3}\frac{h\omega}{2\pi}$$

**Problem 4.**

A simple model for the contribution of phonons to the energy of a solid is the energy of an ideal gas of phonons with an upper limit $\epsilon_{\max}$ on the energy of the orbital:

$$U = \frac{4\pi V}{h^3 c_s^3} \int_0^{\epsilon_{\max}} d\epsilon \, \frac{\epsilon^3}{e^{\beta\epsilon} - 1},$$

where $c_s$ is the speed of sound in the solid.

(A) Why is it appropriate to set the chemical potential equal to 0?

phonons can be absorbed and emitted by the atoms of the solid
chemical equilibrium under atom + phonon $\leftrightarrow$ atom
requires $\mu_{phonon} = 0$

(B) What would the model predict for the number of phonons?

$$N = \frac{4\pi V}{h^3 c_s^3} \int_0^{\epsilon_{max}} d\epsilon \, \frac{\epsilon^2}{e^{\beta\epsilon} - 1}$$

(C) Calculate $U$ and the ~~specific~~ capacity heat of the phonons in the low-temperature limit $kT \ll \epsilon_{\max}$.

The upper limit of the integral can be extended to $\infty$

$$U = \frac{4\pi V}{h^3 c_s^3} \int_0^\infty d\epsilon \, \frac{\epsilon^3}{e^{\beta\epsilon} - 1} = \frac{4\pi V}{h^3 c_s^3} \frac{1}{\beta^4} \int_0^\infty dt \, \frac{t^3}{e^t - 1} = aT^4$$

$$C = \frac{\partial U}{\partial T} = 4aT^3 \qquad a = \frac{4\pi V k^4}{h^3 c_s^3} \int_0^\infty dt \, \frac{t^3}{e^t - 1}$$

(D) Calculate $U$ and the ~~specific~~ capacity heat of the phonons in the high-temperature limit $kT \gg \epsilon_{\max}$.

The Bose-Einstein distribution can be approximated by $1 + \beta\epsilon$

$$U = \frac{4\pi V}{h^3 c_s^3} \int_0^{\epsilon_{max}} d\epsilon \, \frac{\epsilon^3}{\beta\epsilon} = \frac{4\pi V}{h^3 c_s^3} \frac{1}{\beta} \int_0^{\epsilon_{max}} d\epsilon \, \epsilon^2 = \frac{4\pi V}{h^3 c_s^3} kT \, \frac{\epsilon_{max}^3}{3}$$

(E) In the Debye model, phonons are assumed to account for all the specific heat of the solid. If the solid is a 3-dimensional lattice of $M$ atoms, what is its specific heat in the high-temperature limit? Write down the equation that determines $\epsilon_{\max}$ in the Debye model.

vibrations of M atoms in 3 dimensions
$\Longrightarrow$ 6M quadratic degrees of freedom

equipartition theorem $\Longrightarrow$ $U = 6M \cdot \frac{1}{2} kT = 3MkT$

Debye model: $\displaystyle \int \frac{d^3r \, d^3p}{h^3} \bigg|_{\epsilon < \epsilon_{max}} = 6M$

$$\frac{V}{h^3} 4\pi \int_0^{\epsilon_{max}/c_s} p^2 dp = 6M$$

$$\frac{4\pi V}{h^3} \frac{1}{3} \left(\frac{\epsilon_{max}}{c_s}\right)^3 = 6M$$