Winter 2012

Physics 622 1st Midterm Exam NAME ______________________

Problem 1.

An ideal paramagnet consists of N spins with magnetic moment μ in a magnetic field B at temperature T. The Helmholtz free energy for this system is

$$F = -NkT \ln[2\cosh(\beta\mu B)],$$

where $\beta = 1/kT$. The thermodynamic relation for F is

$$dF = -SdT - MdB.$$

The following derivative might be useful:

$$\frac{d}{dx}\ln[2\cosh x] = \tanh x.$$

(A) What is the partition function Z for this system?

$$Z = e^{-\beta F} = \left[2\cosh(\mu B/kT)\right]^N$$

(B) Express the energy U as a derivative of F and calculate U explicitly.

$$U = -\frac{\partial}{\partial\beta}\log Z = -\frac{\partial}{\partial\beta}(-\beta F)$$

$$= -\frac{\partial}{\partial\beta}\left(N\ln[2\cosh(\beta\mu B)]\right)$$

$$= -N\tanh(\beta\mu B)\cdot\mu B$$

$$= -N\mu B\tanh(\mu B/kT)$$

(C) Express the entropy S as a derivative of F and calculate ~~U~~ S explicitly.

$$S = -\left(\frac{\partial F}{\partial T}\right)_B$$

$$= +Nk\ln[2\cosh(\mu B/kT)] + NkT\tanh\frac{\mu B}{kT}\left(-\frac{\mu B}{kT^2}\right)$$

$$= Nk\left(\ln[2\cosh(\mu B/kT) - \frac{\mu B}{kT}\tanh\frac{\mu B}{kT}\right)$$

Consider a single spin in equilibrium at temperature T. Its energy is $E = -\mu B$ if the spin is up ($\uparrow$) and $E = +\mu B$ if the spin is down ($\downarrow$).

(D) Express the partition function Z_1 for the single spin as a sum of Boltzmann factors.

$$Z_1 = e^{-\beta(-\mu B)} + e^{-\beta(+\mu B)} = e^{\beta\mu B} + e^{-\beta\mu B}$$

(E) Express the partition function Z for the entire system of N spins in terms of Z_1.

$$Z = Z_1^N$$

(F) What is the probability $\mathcal{P}_\uparrow$ that the single spin is up?

$$\mathcal{P}_\uparrow = \frac{e^{\beta\mu B}}{e^{\beta\mu B} + e^{-\beta\mu B}}$$

(G) What is the average energy $\bar{E}$ of the single spin? (Express it using the probabilities $\mathcal{P}_\uparrow$ and $\mathcal{P}_\downarrow$ and also as an explicit function of B.)

$$\bar{E} = \mathcal{P}_\uparrow(-\mu B) + \mathcal{P}_\uparrow(+\mu B)$$

$$= \frac{e^{\beta\mu B}}{e^{\beta\mu B} + e^{-\mu B}}(-\mu B) + \frac{e^{-\beta\mu B}}{e^{\beta\mu B} + e^{-\beta\mu B}}(+\mu B)$$

$$= -\mu B\,\frac{e^{\beta\mu B} - e^{-\beta\mu B}}{e^{\beta\mu B} + e^{-\beta\mu B}}$$

(H) Given $\bar{E}$, what is the total energy U of the system?

$$U = N\bar{E}$$

The general definition of the entropy S for a probability distribution $\mathcal{P}(s)$ is

$$S = -\sum_s \mathcal{P}(s)\log\mathcal{P}(s).$$

(I) What is the entropy S_1 for the single spin. (Express it in terms of the probabilities $\mathcal{P}_\uparrow$ and $\mathcal{P}_\downarrow$).

$$S = -\mathcal{P}_\uparrow\log\mathcal{P}_\uparrow - \mathcal{P}_\downarrow\log\mathcal{P}_\downarrow$$

Problem 2.

Neutrinos have a single interacting spin state, but they come in three flavors (e, μ, and τ), which can be treated like 3 spin states. For an ideal gas of N massless neutrinos with equal numbers of the 3 flavors, the number N of neutrinos and their total energy U are

$$N = \frac{12\pi V}{h^3}\int_0^\infty dp\, p^2 \frac{1}{\exp[\beta(pc-\mu)]+1},$$
$$U = \frac{12\pi V c}{h^3}\int_0^\infty dp\, p^3 \frac{1}{\exp[\beta(pc-\mu)]+1}.$$

First consider the system at the absolute zero of temperature.

(A) Evaluate the two integrals analytically.

$$N = \frac{12\pi V}{h^3}\int_0^{p_F} dp\, p^2 = \frac{12\pi V}{h^3}\frac{p_F^3}{3} = \frac{4\pi V p_F^3}{h^3}$$

where $p_F = \mu/c$

$$U = \frac{12\pi V c}{h^3}\int_0^{p_F} dp\, p^3 = \frac{12\pi V c}{h^3}\cdot\frac{p_F^4}{4} = \frac{3\pi V c\, p_F^4}{h^3}$$

(B) Solve for the chemical potential μ as a function of N.

$$\mu = p_F c = \left(\frac{N h^3}{4\pi V}\right)^{1/3} c = \left(\frac{N}{4\pi V}\right)^{1/3} hc$$

(C) Express the energy U as a function of N and V only.

$$U = \frac{3\pi V c}{h^3}\left[\left(\frac{N}{4\pi V}\right)^{1/3} h\right]^4 = 3\pi V\left(\frac{N}{4\pi V}\right)^{4/3} hc$$
$$= \frac{3}{4}N\left(\frac{N}{4\pi V}\right)^{1/3} hc$$

Next consider the system at very low temperature. The integrals for N and U can be calculated as expansions in kT/μ:

$$N = N\left(\frac{\mu}{\varepsilon_F}\right)^3\left[1+\pi^2\frac{kT}{\mu}+\ldots\right],$$
$$U = \frac{3N}{4}\varepsilon_F\left(\frac{\mu}{\varepsilon_F}\right)^4\left[1+2\pi^2\frac{kT}{\mu}+\ldots\right],$$

where ε_F is the Fermi energy. The solution for μ and the expression for U can be expanded in powers of kT/ε_F.

(D) Solve for μ to first order in kT/ε_F.

$$1 = \left(\frac{\mu}{\epsilon_F}\right)^3\left[1+\pi^2\frac{kT}{\epsilon_F}+\cdots\right]$$

$$\mu = \epsilon_F\left[1+\pi^2\frac{kT}{\epsilon_F}+\cdots\right]^{-1/3} = \epsilon_F\left[1-\frac{1}{3}\pi^2\frac{kT}{\epsilon_F}+\cdots\right]$$

Finally consider the system at very high temperature. The integrals for N and U can be calculated as expansions in $\exp(\beta\mu)$:

$$N = e^{\beta\mu} 16\pi V \left(\frac{kT}{hc}\right)^3 \left[1 - \frac{1}{8}e^{\beta\mu} + \ldots\right],$$
$$U = e^{\beta\mu} 48\pi V hc \left(\frac{kT}{hc}\right)^4 \left[1 - \frac{1}{16}e^{\beta\mu} + \ldots\right].$$

The solution for μ and the expression for U can be expanded in powers of $Y \equiv (N/V)(hc/kT)^3$.

(E) Solve for $\exp(\beta\mu)$ at leading order in Y. (At this order, the expressions in square brackets can be set equal to 1.)

$$N = e^{\beta\mu} 16\pi V \left(\frac{kT}{hc}\right)^3$$

$$e^{\beta\mu} = \frac{N}{16\pi V}\left(\frac{hc}{kT}\right)^3$$

(F) Determine the energy U of the classical gas of neutrinos.

$$U = e^{\beta\mu} 48\pi V hc \left(\frac{kT}{hc}\right)^4$$
$$= \frac{N}{16\pi V}\left(\frac{hc}{kT}\right)^3 48\pi V hc \left(\frac{kT}{hc}\right)^4 = 3NkT$$

(G) The solution for $\exp(\beta\mu)$ at next-to-leading order in Y is

$$e^{\beta\mu} = \frac{1}{16\pi}Y\left(1 + \frac{1}{128\pi}Y\right).$$

Calculate U to next-to-leading order in Y.

$$U = \frac{1}{16\pi}Y\left(1+\frac{1}{128\pi}Y\right) 48\pi V hc \left(\frac{kT}{hc}\right)^4 \left[1 - \frac{1}{16}\frac{1}{16\pi}Y\right]$$
$$= 3\,Y\,Vhc\left(\frac{kT}{hc}\right)^4\left[1 + \frac{1}{128\pi}Y - \frac{1}{256\pi}Y\right]$$
$$= 3NkT\left[1 + \frac{1}{256\pi}Y\right]$$

(H) The typical momentum of a particle in the gas is of order kT/c. Show that the condition $Y \ll 1$ is equivalent to the condition that the cube of the de Broglie wavelength h/p of a typical particle is small compared to the volume per particle.

$$Y << 1 \implies \frac{N}{V}\left(\frac{hc}{kT}\right)^3 << 1$$

$$\left(\frac{h}{(kT/c)}\right)^3 << \frac{V}{N} \implies \frac{N}{V}\left(\frac{hc}{kT}\right)^3 << 1 \qquad \text{same condition!}$$

Problem 3.

A system consists of a gas of N particles in a volume V at temperature T. Each of the following specifies a possible probability distribution $\mathcal{P}(s)$ for the microstates of the system in terms of the energy U_s of the microstate s and its particle number N_s:

- **R**: $\mathcal{P}(s) \propto \exp[S_{\text{res}}(U_{\text{total}} - U_s)]$, where $S_{\text{res}}(U)$ is the entropy of a much larger reservoir as a function of its energy U and U_{total} is the conserved total energy of the reservoir plus the system,

- **C**: $\mathcal{P}(s) \propto \exp[-\beta U_s]$, where β is a parameter,

- **G**: $\mathcal{P}(s) \propto \exp[-\beta(U_s - \mu N_s)]$, where β and μ are parameters.

(A) For **R**, specify the constraint on the entropy function $S_{\text{res}}(U)$ that is required in order to describe the system, which has temperature T.

$$\frac{\partial S_{res}}{\partial U}(U_{total}) = \frac{1}{T}$$

(B) For **C**, identify the parameter β.

$$\beta = \frac{1}{kT}$$

(C) For **G**, identify the parameter β. Write down the equation that determines the parameter μ.

$$\beta = \frac{1}{kT}$$

$$\bar{N} = N$$

(D) For **C**, express the average energy $\bar{U}$ of the system as a ratio of sums over microstates.

$$\bar{U} = \frac{\sum_s U_s e^{-\beta U_s}}{\sum_s e^{-\beta U_s}}$$

(E) For **G**, express the average particle number $\bar{N}$ of the system as a ratio of sums over microstates.

$$\bar{N} = \frac{\sum_s N_s e^{-\beta(U_s - \mu N_s)}}{\sum_s e^{-\beta(U_s - \mu N_s)}}$$

(F) Specify the thermodynamic limit of the system in terms of the variables N, V, and T.

$$N \to \infty \qquad \frac{N}{V} \to \infty$$
$$V \to \infty$$

(G) Of the three probability distributions (**R**, **C**, and **G**), which ones can correctly describe the thermodynamic limit of the system.

R, G, and G

Recall that the system consists of exactly N particles in the volume V. Suppose the temperature T of the system is held fixed by keeping it in thermal contact with an infinitely large reservoir with that temperature.

(H) How does the relative fluctuation $\Delta U/\bar{U}$ in the energy of the system scale with N?

$$\frac{\Delta U}{\bar{U}} \sim \frac{1}{\sqrt{N}}$$

(I) What is the average particle number $\bar{N}$ for the system? What is the standard deviation ΔN in the particle number?

$$\bar{N} = N$$
$$\Delta N = 0$$

(J) Of the three probability distributions (**R**, **C**, and **G**), which one describes the system most accurately, including fluctuations in the thermodynamic variables?

C