Physics 622 2nd Midterm Exam NAME ____________________

Problem 1.

A large number N of identical bosonic atoms, all in the same spin state, are confined inside a cube of length L. The orbitals are labelled by quantum numbers n_x, n_y, n_z with values 1,2,3,.... The orbital energies are $\varepsilon_{n_x,n_y,n_z} = (n_x^2 + n_y^2 + n_z^2)h^2/(8mL^2)$.

(A) What is the lowest-energy orbital and what is its energy ε_0?

$n_x = 1, n_y = 1, n_z = 1$

$$\epsilon_0 = 3\frac{h^2}{8mL^2}$$

(B) Describe the ground state of the system of N atoms. What is its total energy U?

all N atoms in the lowest energy orbital: $n_x=1, n_y=1, n_z=1$

$$U = N\epsilon_0 = N\frac{3h^2}{8mL^2}$$

Suppose the atoms are in equilibrium at temperature T and chemical potential μ.

(C) What is the average number N_0 of atoms in the lowest-energy orbital.

$$N_0 = \frac{1}{e^{\beta(\epsilon_0-\mu)}-1}, \quad \beta = \frac{1}{kT}$$

(D) Express the average total number $\bar{N}$ of atoms as a sum over quantum numbers (with definite upper and lower limits).

$$\bar{N} = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} \frac{1}{e^{\beta(\epsilon_{n_x n_y n_z}-\mu)}-1}$$

(E) Give the best possible upper bound on the chemical potential μ. What is this upper bound in the thermodynamic limit?

$\mu < \varepsilon_0$

thermodynamic limit: $L \to \infty \Rightarrow \varepsilon_0 \to 0$

$\mu \leq 0$

(F) Define the thermodynamic limit in terms of the variables N, L, and T.

$N \to \infty$
$L \to \infty$ with $\frac{N}{L^3}$ fixed

T fixed

Now focus on the thermodynamic limit for this system of N identical bosonic atoms, all in the same spin state, confined to a volume $V = L^3$.

(G) Express the orbital energy ε as a function of the momentum.

$$\epsilon = \frac{p^2}{2m}$$

(H) Determine the critical temperature T_c for Bose-Einstein condensation as a function of the number density N/V. (The integrals at the bottom of the page may be useful.)

$$N = \frac{V}{h^3} \int d^3p \, \frac{1}{e^{\beta p^2/2m} - 1} = \frac{V}{h^3} \zeta(\tfrac{3}{2}) (2\pi m k T)^{3/2}$$

$$kT_c = \frac{h^2}{2\pi m} \left(\frac{1}{\zeta(3/2)} \frac{N}{V} \right)^{2/3}$$

(I) For temperatures $T > T_c$, express N and the total energy U in terms of momentum integrals.

$$N = \frac{V}{h^3} \int d^3p \, \frac{1}{e^{\beta(\epsilon - \mu)} - 1} \qquad \epsilon = \frac{p^2}{2m}$$

$$U = \frac{V}{h^3} \int d^3p \, \frac{\epsilon}{e^{\beta(\epsilon - \mu)} - 1}$$

(J) For temperatures $T < T_c$, express N and U in terms of momentum integrals (with μ eliminated in favor of N_0).

$$N = N_0 + \frac{V}{h^3} \int d^3p \, \frac{1}{e^{\beta\epsilon} - 1} \qquad \epsilon = \frac{p^2}{2m}$$

$$U = N_0 \epsilon_0 + \frac{V}{h^3} \int d^3p \, \frac{\epsilon}{e^{\beta\epsilon} - 1}$$

(K) For $T < T_c$, determine N_0 as a function of T. Express the condensate fraction N_0/N as a function of T/T_c only.

$$N = N_0 + \frac{V}{h^3} \zeta(\tfrac{3}{2}) (2\pi m kT)^{3/2} \qquad N_0 = N - \frac{V}{h^3} \zeta(\tfrac{3}{2}) (2\pi m kT)^{3/2}$$

divide by $N = \frac{V}{h^3} (2\pi m k T_c)^{3/2}$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c} \right)^{3/2}$$

(L) For $T < T_c$, determine U as a function of T.

$$U = N_0 \epsilon_0 + \frac{V}{h^3} \frac{3}{2\pi} \zeta(\tfrac{5}{2}) (2\pi m kT)^{5/2}$$

$$\int d^3p \frac{1}{\exp(p^2/2mkT) - 1} = \zeta(\tfrac{3}{2})(2\pi mkT)^{3/2}, \quad \int d^3p \frac{p^2}{\exp(p^2/2mkT) - 1} = \frac{3}{2\pi}\zeta(\tfrac{5}{2})(2\pi mkT)^{5/2}$$

Problem 2.

The differential power in electromagnetic radiation emitted by a black body of surface area A and temperature T is

$$d\mathcal{P} = A\frac{2\pi h}{c^2}\frac{f^3}{\exp(hf/kT)-1}df.$$

(A) Stefan's Law states that the total power radiated by the black body is $\mathcal{P} = \sigma T^4 A$, where σ is a constant. Express σ in terms of physical constants and an integral over a dimensionless variable x.

$$\mathcal{P} = A\frac{2\pi h}{c^2}\int_0^\infty df\,\frac{f^3}{e^{hf/kT}-1} = A\frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4\int_0^\infty dx\,\frac{x^3}{e^x-1}$$

$$\sigma = \frac{2\pi k^4}{c^2h^3}\int_0^\infty dx\,\frac{x^3}{e^x-1}$$

(B) Wien's Law states that the frequency $f_{\max}$ at which the maximum power is radiated is proportional to T. The law can be expressed as $f_{\max} = xkT/h$, where x is a numerical constant. Derive an equation for x.

maximum power $\Rightarrow$ $\frac{d}{df}\frac{2\pi h}{c^2}\frac{f^3}{e^{hf/kT}-1} = 0$

$$\frac{3f^2}{e^{hf/kT}-1} - \frac{f^3}{(e^{hf/kT}-1)^2}e^{hf/kT}\frac{h}{kT} = 0$$

multiply by $(h/kT)^2$: $\frac{3x^2}{e^x-1} - \frac{x^3e^x}{(e^x-1)^2} = 0$

The electromagnetic energy inside a cavity of volume V in a conductor with temperature T can be expressed as an integral over the photon energy ε:

$$U = V\frac{8\pi}{h^3c^3}\int_0^\infty\frac{\varepsilon^3}{\exp(\varepsilon/kT)-1}d\varepsilon.$$

(C) State as precisely as possible the relation between the shape of the frequency spectrum emitted by a black body of temperature T and the shape of the photon energy distribution in a spherical cavity at the same temperature T.

frequency spectrum has same shape as energy distribution if we set $f = \varepsilon/h$

(D) Suppose the black body is a sphere of radius R and the cavity is also a sphere of radius R. State as precisely as possible the relation between the total power $\mathcal{P}$ emitted by the black body and the total energy U in the cavity.

$$\mathcal{P} = 4\pi R^2\cdot\frac{2\pi}{c^2h^3}(kT)^4\int_0^\infty dx\,\frac{x^3}{e^x-1}$$

$$U = \frac{4}{3}\pi R^3\cdot\frac{8\pi}{h^3c^3}(kT)^4\int_0^\infty dx\,\frac{x^3}{e^x-1}$$

$$\mathcal{P} = \frac{3c}{4R}U$$

(E) Suppose a small cylindrical hole of radius r is drilled into the conductor so that electromagnetic radiation can leak out from the cavity through the hole. What is the power radiated from the hole?

$$\mathcal{P} = \sigma T^4\cdot(\pi r^2)$$

In a 2-dimensional universe, electromagnetic (EM) waves have only one tranverse polarization mode. Consider a 2-dimensional cavity in a conductor in such a universe.

If the cavity is a square of area L^2, the normal modes of standing EM waves can be labelled by integers $n_x, n_y = 1, 2, 3, \ldots$. The amplitude $a(t)$ of a normal mode oscillates with frequency $f = \sqrt{n_x^2 + n_y^2}\, c/2L$.

(E) Suppose the amplitude of each normal mode can be treated as a classical harmonic oscillator that is in equilibrium at temperature T. Use the equipartition theorem to determine the average energy $\bar{E}$ in a single standing wave of frequency f.

1 oscillator $\Rightarrow$ 2 quadratic degrees of freedom

equipartition theorem: $\bar{E} = 2\left(\tfrac{1}{2}kT\right) = kT$

(F) Use the equipartition theorem to determine the average total energy $\bar{U}$ in all the EM standing waves. Is there anything unphysical about the result for $\bar{U}$?

$$\bar{U} = \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} kT = \infty \cdot kT$$

ultraviolet catastrophe: $\bar{U}$ is divergent

(G) Suppose the energy in a oscillator of frequency f is quantized with quantum hf: $E = nhf$, $n = 0, 1, 2, \ldots$. If the oscillator is in equilibrium at temperature T, what is its average energy $\bar{E}$?

$$\bar{E} = \frac{\sum_{n=0}^{\infty} nhf\, e^{-\beta nhf}}{\sum_{n=0}^{\infty} e^{-\beta nhf}} = \frac{hf}{e^{\beta hf} - 1}$$

(H) Under the same assumptions, what is the average total energy $\bar{U}$ in all the EM standing waves?

$$\bar{U} = \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \frac{h\sqrt{n_x^2+n_y^2}\, c/2L}{e^{\beta h\sqrt{n_x^2+n_y^2}\, c/2L} - 1}$$

Now consider this system in the thermodynamic limit. Suppose the electromagnetic energy is in the form of photons, which are massless bosons with energy-momentum relation $\varepsilon = \sqrt{p_x^2 + p_y^2}\, c$ and chemical potential 0.

(I) What is the average energy per area?

$$U = \frac{A}{h^2} \int d^2p\, \frac{\epsilon}{e^{\beta\epsilon} - 1}, \quad \epsilon = pc \qquad \frac{U}{A} = \frac{2\pi}{h^2} \int_0^{\infty} p\,dp\, \frac{pc}{e^{\beta pc} - 1}$$

(J) What is the average number of photons per area?

$$N_{photons} = \frac{A}{h^2} \int d^3p\, \frac{1}{e^{\beta\epsilon} - 1}, \quad \epsilon = pc \qquad \frac{N_{photon}}{A} = \frac{2\pi}{h^2} \int_0^{\infty} p\,dp\, \frac{1}{e^{\beta pc} - 1}$$

Problem 3.

A single orbital with orbital energy ε is in both thermal and chemical equilibrium with a reservoir of temperature T and chemical potential μ.

Suppose the only possible occupation numbers n of the orbital are 0 and 1.

(A) What is the ground state partition function $\mathcal{Z}_1$ for the single orbital?

$$\mathcal{Z}_1 = \sum_{n=0}^{1} e^{-\beta(\epsilon-\mu)n} = 1 + e^{-\beta(\epsilon-\mu)}$$

(B) What are the probabilities $\mathcal{P}(0)$ and $\mathcal{P}(1)$ for the orbital to have occupation numbers 0 and 1?

$$\mathcal{P}(0) = \frac{1}{1+e^{-\beta(\epsilon-\mu)}} \qquad \mathcal{P}(1) = \frac{e^{-\beta(\epsilon-\mu)}}{1+e^{-\beta(\epsilon-\mu)}}$$

(C) What is the average occupation number $\bar{n}$?

$$\bar{n} = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

(D) What is the average energy $\bar{E}$ of the orbital?

$$\bar{E} = \frac{\epsilon}{e^{\beta(\epsilon-\mu)}+1}$$

Suppose the occupation number n of the orbital can be any nonnegative integer: 0,1,2,3,. . . .

(E) What is the ground state partition function $\mathcal{Z}_1$ for the single orbital?

$$\mathcal{Z}_1 = \sum_{n=0}^{\infty} e^{-\beta(\epsilon-\mu)n} = \frac{1}{1-e^{-\beta(\epsilon-\mu)}}$$

(F) What is the probability distribution $\mathcal{P}(n)$ for the occupation number n?

$$\mathcal{P}(n) = \frac{1}{\mathcal{Z}_1} e^{-\beta(\epsilon-\mu)n}$$

(G) What is the average occupation number $\bar{n}$?

$$\bar{n} = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

(H) What is the energy E in the orbital if its occupation number is n? What is the average energy $\bar{E}$ of the orbital?

$$E = n\varepsilon$$

$$\bar{E} = \frac{\epsilon}{e^{\beta(\epsilon-\mu)}-1}$$

A system consists of many identical fermions, all in the same spin state, with temperature T and chemical potential μ. The fermions have K distinct orbitals labelled by $k = 1, 2, \ldots, K$ with energies ε_1, ε_2, ..., ε_K. Express each of the following in terms of a sum over the orbitals.

(I) the average number $\bar{N}$ of fermions.

$$\bar{N} = \sum_{k=1}^{K} \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

(J) the average total energy $\bar{U}$.

$$\bar{U} = \sum_{k=1}^{K} \frac{\epsilon_k}{e^{\beta(\epsilon_k - \mu)} + 1}$$

(K) the logarithm of the grand partition function $\mathcal{Z}$.

$$\log \mathcal{Z} = \sum_{k=1}^{K} \log\left(1 + e^{-\beta(\epsilon_k - \mu)}\right)$$

Suppose the system consists of exactly N identical noninteracting fermions, in a volume V all in the same spin state, at temperature T. In the thermodynamic limit, it can be described using the grand canonical ensemble with temperature T and some chemical potential μ.

(K) Express the orbital energy as a function of the momentum of the fermion. Express the sum over orbitals as a momentum integral.

$$\epsilon = \frac{p^2}{2m}$$

$$\sum_{\text{orbital}} = \frac{V}{h^3} \int d^3p$$

(L) Write down the equation whose solution determines the chemical potential μ. (It should involve a sum over orbitals.)

$$N = \bar{N}$$

$$N = \frac{V}{h^3} \int d^3p \, \frac{1}{e^{\beta(p^2/2m)} + 1}$$