Quiz # 1 NAME ______________________________

An ideal gas consists of a large number $N$ of identical ultrarelativistic electrons confined to a thin planar region of area $A$. Use the grand canonical ensemble with chemical potential $\mu = \varepsilon_F$ and temperature 0 to derive its ground state energy $U$.

(a) Express the sum over orbitals in terms of integrals over position vectors $\boldsymbol{r} = (x, y)$ and momentum vectors $\boldsymbol{p} = (p_x, p_y)$.
Express the sum over orbitals in terms of an integral over the orbital energy $\varepsilon = pc$.

$$\sum_i = 2\,\frac{\int d^2r \int d^2p}{h^2} = \frac{2A}{h^2}\int 2\pi p\,dp$$

$$= \frac{4\pi A}{h^2}\int \frac{\epsilon}{c}\,\frac{d\epsilon}{c} = \frac{4\pi A}{h^2c^2}\int \epsilon\, d\epsilon$$

(b) Express $N$ and $U$ as definite integrals (with upper and lower limits) over $\varepsilon$.

$$N = \frac{4\pi A}{h^2c^2}\int_0^{\epsilon_F} \epsilon\, d\epsilon$$

$$U = \frac{4\pi A}{h^2c^2}\int_0^{\epsilon_F} \epsilon^2\, d\epsilon$$

(c) Evaluate the integrals over $\varepsilon$.

$$\int_0^{\epsilon_F} d\epsilon = \frac{\epsilon^2}{2}\Big|_0^{\epsilon_F} = \frac{1}{2}\epsilon_F^2$$

$$\int_0^{\epsilon_F} \epsilon^2 d\epsilon = \frac{\epsilon^3}{3}\Big|_0^{\epsilon_F} = \frac{1}{3}\epsilon_F^3$$

(d) Eliminate the chemical potential to get an expression for $U$ as a function of $N$ and $A$.

$$N = \frac{2\pi A}{h^2c^2}\epsilon_F^2 \implies \epsilon_F = \left(\frac{h^2c^2}{2\pi}\frac{N}{A}\right)^{1/2} = hc\sqrt{\frac{N}{2\pi A}}$$

$$U = \frac{4\pi A}{3h^2c^2}\epsilon_F^3 = \frac{4\pi A}{3h^2c^2}\left(hc\sqrt{\frac{N}{2\pi A}}\right)^3$$

$$= \frac{2}{3}Nhc\sqrt{\frac{N}{2\pi A}} = \frac{2}{3}N\epsilon_F$$