Quiz # 3 NAME ______________________________

An ideal gas consists of N massless identical bosons (with energy-momentum relation $\varepsilon = pc$), all in the same spin state, in a volume V at temperature T.

(a) For $T > T_c$, express N as an integral that depends on T and the chemical potential μ.

$$N = \frac{V}{h^3}\int d^3p \frac{1}{e^{\beta(\epsilon-\mu)}-1} = \frac{V}{h^3} 4\pi \int_0^\infty p^2 dp \frac{1}{e^{\beta(pc-\mu)}-1}$$

$$= \frac{4\pi V}{h^3 c^3}\int_0^\infty d\epsilon \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1}$$

(b) For $T < T_c$, express N in terms of the condensate number N_0 and an integral that depends on T.

$$N = N_0 + \frac{V}{h^3}\int d^3p \frac{1}{e^{\beta\epsilon}-1}$$

$$= N_0 + \frac{4\pi V}{h^3 c^3}\int_0^\infty d\epsilon \frac{\epsilon^2}{e^{\beta\epsilon}-1}$$

(c) At the phase transition for Bose-Einstein condensation, $T = T_c$, $\mu = 0$ and $N_0 = 0$. Solve for the critical temperature T_c as a function of N/V.

$$N = \frac{4\pi V}{h^3 c^3}\int_0^\infty d\epsilon \frac{\epsilon^2}{e^{\beta\epsilon}-1} = \frac{4\pi V}{h^3 c^3}\frac{1}{\beta^3}\int_0^\infty dx \frac{x^2}{e^x-1}$$

$$= \frac{4\pi V}{h^3 c^3}(kT)^3 \cdot 2\zeta(3)$$

$$kT_c = hc\left(\frac{1}{8\pi\zeta(3)}\frac{N}{V}\right)^{1/3}$$

integral table:

$$\int_0^\infty dx \frac{x}{e^x-1} = \frac{\pi^2}{6}, \qquad \int_0^\infty dx \frac{x^2}{e^x-1} = 2\zeta(3). \qquad \int_0^\infty dx \frac{x^3}{e^x-1} = \frac{\pi^4}{15}.$$