Therefore the Fermi energy is

$$\epsilon_F = \frac{h^2}{8mL^2}\left(\frac{3N}{2\pi}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3N}{2\pi V}\right)^{2/3}.$$

Plugging in the numbers for nuclear matter gives

$$\epsilon_F = \frac{(6.63\times10^{-34}\ \text{J·s})^2}{8(1.67\times10^{-27}\ \text{kg})}\left(\frac{3(0.18\times10^{45}\ \text{m}^{-3})}{2\pi}\right)^{2/3} = 6.4\times10^{-12}\ \text{J} = 40\ \text{MeV}.$$

The Fermi temperature is just this divided by Boltzmann's constant:

$$T_F = \frac{\epsilon_F}{k} = \frac{4\times10^7\ \text{eV}}{8.62\times10^{-5}\ \text{eV/K}} = 4.6\times10^{11}\ \text{K}.$$

This is hotter than the center of any ordinary star. Therefore, to the extent that such a small system can be treated using thermodynamics at all, it should be an excellent approximation to treat a nucleus as a degenerate, $T = 0$ Fermi gas, in virtually all circumstances. (Exceptions would include heavy ion collisions, supernova explosions, and the very early universe.)

Problem 7.22. (Relativistic Fermi gas at $T = 0$.)

(a) The allowed wavelengths are the same as for a nonrelativistic particle: If the length of the box in the x direction is L, then the allowed wavelengths in the x direction are $\lambda_x = 2L/n_x$, and similarly for λ_y and λ_z. The momenta are also the same: $p_x = hn_x/2L$, and similarly for p_y and p_z. But now the energy is

$$\epsilon = pc = c\sqrt{p_x^2+p_y^2+p_z^2} = \frac{hc}{2L}\sqrt{n_x^2+n_y^2+n_z^2} = \frac{hcn}{2L},$$

where $n = \sqrt{n_x^2+n_y^2+n_z^2}$. Each of the n's can be any positive integer, so we can visualize the single-particle states as a lattice of points in the first octant of n-space. As in the nonrelativistic case, the energy of a state depends only on its distance from the origin, so at $T = 0$ we simply fill up an eighth-sphere, working our way outward to some maximum radius n_{max}. The total number of electrons is just the volume of this eighth-sphere times 2 (since there are two spin states for each set of n's):

$$N = 2\cdot\frac{1}{8}\cdot\frac{4}{3}\pi n_{\text{max}}^3 = \frac{\pi}{3}n_{\text{max}}^3.$$

Solving for n_{max} gives $n_{\text{max}} = (3N/\pi)^{1/3}$. The chemical potential or Fermi energy is just the energy of the last state filled, that is, the energy corresponding to $n = n_{\text{max}}$:

$$\mu = \epsilon_F = \epsilon(n_{\text{max}}) = \frac{hcn_{\text{max}}}{2L} = \frac{hc}{2L}\left(\frac{3N}{\pi}\right)^{1/3} = \frac{hc}{2}\left(\frac{3N}{\pi V}\right)^{1/3}$$

(b) The total energy is the sum of the energies of all the occupied states:

$$U = 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\epsilon(n),$$

where the factor of 2 is for the two spin orientations. As in the nonrelativistic case, convert this sum to an integral in spherical coordinates, being sure to include the "measure" $n^2 \sin\theta$:

$$U = 2\int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \,\sin\theta \int_0^{n_{\max}} dn\, n^2 \cdot \frac{hcn}{2L}.$$

The angular integrals give $\pi/2$ (the surface area of a unit-radius eighth-sphere), leaving us with

$$U = \pi \cdot \frac{hc}{2L}\int_0^{n_{\max}} n^3\, dn = \frac{\pi hc}{2L}\cdot\frac{1}{4}n_{\max}^4 = \frac{\pi hc}{8L}\left(\frac{3N}{\pi}\right)^{4/3} = \frac{3Nhc}{8}\left(\frac{3N}{\pi V}\right)^{1/3} = \frac{3}{4}N\epsilon_F,$$

where ϵ_F is given by the result of part (a). Thus the average energy is 3/4 of the maximum energy, as compared to 3/5 in the nonrelativistic case.

Problem 7.23. (White dwarf stars.)

(a) We want to make something with units of energy (newton-meters) out of M (kg), R (m), and G (N·m²/kg²). It's convenient to express all the units as I just have, taking the three basic units to be newtons, kilograms, and meters; none of these three can be written in terms of the other two. How to do it? Well, to get N in the numerator we need exactly one power of G. But then, to cancel the kg² in the denominator we need two powers of M. And, since G has m² in the numerator and we want just meters, we need to divide by one power of R. Finally, we should put in a minus sign since gravity is attractive: We would have to *add* energy to disassemble the sphere, moving the parts infinitely far apart where they have zero potential energy.

Just for fun, let me now derive the exact formula for the potential energy of a sphere of uniform density ρ. Imagine assembling the sphere by bringing in concentric shells of mass, one at a time, from infinite distance. Suppose, further, that we already have a sphere of radius r and mass $m = 4\pi r^3\rho/3$. We now bring in the next shell, whose thickness is dr and whose mass is therefore $dm = 4\pi r^2 dr \cdot \rho$. The potential energy of this shell once it arrives is $dU = -Gm\,dm/r$. Summing over all such shells and converting the sum to an integral, we obtain for the total potential energy

$$\begin{aligned} U_{\text{grav}} &= \int dU = -\int \frac{Gm}{r}\,dm = -G\int_0^R \frac{1}{r}\left(\frac{4\pi r^3\rho}{3}\right)(4\pi r^2\rho)\,dr \\ &= -\frac{16\pi^2 G\rho^2}{3}\int_0^R r^4\,dr = -\frac{16\pi^2 G\rho^2}{3}\cdot\frac{R^5}{5} = -\frac{16\pi^2 GR^5}{15}\left(\frac{3M}{4\pi R^3}\right)^2 \\ &= -\frac{3}{5}\frac{GM^2}{R}, \end{aligned}$$

where I've substituted $\rho = 3M/4\pi R^3$ in the second-to-last step. So the numerical coefficient in the energy formula, for the (probably unrealistic) case of a uniform-density sphere, is 3/5.

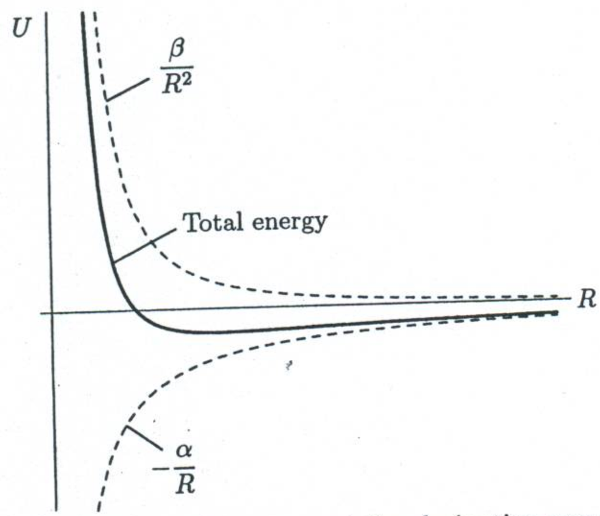
(b) According to equations 7.42 and 7.39, the total energy of a degenerate electron gas is

$$U_{\text{kinetic}} = \frac{3}{5}N\epsilon_F = \frac{3}{5}N \cdot \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{2/3},$$

where N is the number of electrons. If the star contains one proton (mass m_p) and one neutron (mass $\approx m_p$) for each electron, then $N = M/2m_p$. Plugging in $\frac{4}{3}\pi R^3$ for the volume then gives

$$U_{\text{kinetic}} = \frac{3h^2}{40m_e}\left(\frac{M}{2m_p}\right)^{5/3}\left(\frac{9}{4\pi^2 R^3}\right)^{2/3} = (0.0088)\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2}.$$

(c) The gravitational energy of the star is proportional to $-1/R$, while the kinetic energy of the electrons is proportional to $+1/R^2$. Here's a sketch of these functions and their sum:



To find the minimum in the total energy, set the derivative equal to zero:

$$0 = \frac{d}{dR}\left(-\frac{\alpha}{R} + \frac{\beta}{R^2}\right) = \frac{\alpha}{R^2} - \frac{2\beta}{R^3} = \frac{1}{R^2}\left(\alpha - \frac{2\beta}{R}\right).$$

The equilibrium radius is therefore at

$$R = \frac{2\beta}{\alpha} = \frac{2(0.0088)h^2 M^{5/3}/m_e m_p^{5/3}}{(3/5)GM^2} = (0.029)\frac{h^2}{G m_e m_p^{5/3}}\frac{1}{M^{1/3}}.$$

Notice that a white dwarf star with a larger mass has a *smaller* equilibrium radius. This does make sense, because adding mass creates more gravitational attraction, allowing the gravitational energy to decrease more then the kinetic energy increases as the star contracts.

(d) For a one-solar-mass white dwarf,

$$R = \frac{(0.029)(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2}{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(9.11\times10^{-31}\ \text{kg})(1.67\times10^{-27}\ \text{kg})^{5/3}(2\times10^{30}\ \text{kg})^{1/3}}$$

$$= 7.2\times10^6\ \text{m} = 7200\ \text{km}.$$

This is just slightly larger than the earth. (For comparison, the sun's radius is more than 100 times the earth's.) The density is the mass divided by the volume:

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{2 \times 10^{30}\ \text{kg}}{\frac{4}{3}\pi(7.2 \times 10^6\ \text{m})^3} = 1.3 \times 10^9\ \text{kg/m}^3.$$

This is 1.3 million times the density of water.

(e) The Fermi energy is

$$\begin{aligned}\epsilon_F &= \frac{h^2}{8m_e}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{h^2}{8m_e}\left(\frac{9M}{8\pi^2 m_p}\right)^{2/3}\frac{1}{R^2}\\ &= \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11 \times 10^{-31}\ \text{kg})}\left(\frac{9(2 \times 10^{30}\ \text{kg})}{8\pi^2(1.67 \times 10^{-27}\ \text{kg})}\right)^{2/3}\frac{1}{(7.2 \times 10^6\ \text{m})^2}\\ &= 3.1 \times 10^{-14}\ \text{J} = 1.9 \times 10^5\ \text{eV}.\end{aligned}$$

So the Fermi temperature is

$$T_F = \epsilon_F/k = 2.3 \times 10^9\ \text{K}.$$

This is more than a hundred times hotter than the center of the sun. It seems unlikely that the actual temperature of a white dwarf star would be anywhere near this high. In other words, the thermal energy of the electrons is almost certainly much smaller than the kinetic energy they have even at $T = 0$. For the purposes of the energy calculations in this problem, therefore, simply neglecting the thermal energy and setting $T = 0$ is probably an excellent approximation.

(f) If the electrons are ultra-relativistic, we can use the formulas derived in the previous problem for the Fermi energy and the total kinetic energy:

$$\begin{aligned}U_{\text{kinetic}} &= \frac{3}{4}N\epsilon_F = \frac{3}{4}N \cdot \frac{hc}{2}\left(\frac{3N}{\pi V}\right)^{1/3}\\ &= \frac{3}{8}hc\left(\frac{M}{2m_p}\right)^{4/3}\left(\frac{3}{\pi \cdot \frac{4}{3}\pi R^3}\right)^{1/3} = (0.091)hc\left(\frac{M}{m_p}\right)^{4/3}\frac{1}{R}.\end{aligned}$$

The important feature of this formula is that it is proportional to $1/R$, not $1/R^2$. When we add the gravitational potential energy, which is proportional to $-1/R$, we get a total energy function with no stable minimum. Instead, depending on which coefficient is larger, the total energy is simply proportional to either $+1/R$ or $-1/R$. Therefore the "star" will either expand to infinite radius or collapse to zero radius.

(g) First note that the coefficient of the gravitational energy is proportional to M^2, while that of the kinetic energy is proportional to only $M^{4/3}$, so the star will collapse rather than expand if its mass is sufficiently large. The crossover from expansion to collapse occurs when the coefficients are equal, that is, when

$$(0.091)hc\left(\frac{M}{m_p}\right)^{4/3} = \frac{3}{5}GM^2,$$

or

$$M = \left[(0.091)\frac{5}{3}\frac{hc}{G}\right]^{3/2}\frac{1}{m_p^2} = 3.4 \times 10^{30} \text{ kg},$$

that is, a little under twice the sun's mass. However, the star won't be relativistic to begin with unless the average kinetic energy of the electrons is comparable to their rest energy, $mc^2 = 5 \times 10^5$ eV. For the sun's mass, the average electron energy ($0.6\epsilon_F$) is only 1.2×10^5 eV, too low by a factor of about 4.4. This indicates that a one-solar-mass white dwarf is probably stable, but it's still close enough to being relativistic that we shouldn't expect the nonrelativistic approximation to be terribly accurate. Meanwhile, looking back at part (e), we see that the Fermi energy is proportional to $(M/R^3)^{2/3} \propto (M^2)^{2/3} = M^{4/3}$. Therefore, to increase the Fermi energy by a factor of 4.4, we'd have to increase the mass by only a factor of about 3. Conclusion: A white dwarf star with a mass greater than about three times the sun's mass will be relativistic and hence unstable, collapsing to zero radius (unless it first converts into some other form of matter). (Note: The best modern calculations, which take into account both the exact relativistic energy-momentum relation and the variation of density within the star, put the critical mass for a white dwarf at only 1.4 solar masses.)

Problem 7.24. In a neutron star, the kinetic energy comes from the neutrons, and the number of these is simply $N = M/m_n$, where M is the total mass and m_n is the mass of a neutrón. Therefore we can write the kinetic energy as

$$U_{\text{kinetic}} = \frac{3}{5}N\epsilon_F = \frac{3}{5}N \cdot \frac{h^2}{8m_n}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{3h^2}{40m_n}\left(\frac{M}{m_n}\right)^{5/3}\left(\frac{9}{4\pi^2 R^3}\right)^{2/3}.$$

Adding the (negative) gravitational potential energy, we have for the total energy

$$U = U_{\text{potential}} + U_{\text{kinetic}} = -\frac{\alpha}{R} + \frac{\beta}{R^2},$$

where $\alpha = (3/5)GM^2$ and $\beta = (0.028)h^2M^{5/3}/m_n^{8/3}$. As with a white dwarf star, the equilibrium radius is the one that minimizes the total energy. Setting $dU/dR = 0$ and solving for R gives

$$R = \frac{2\beta}{\alpha} = (0.093)\frac{(0.093)h^2}{Gm_n^{8/3}M^{1/3}}.$$

Here again, the equilibrium radius decreases with increasing mass, due to the greater gravitational attraction. For a one-solar-mass neutron star this model predicts,

$$R = \frac{(0.093)(6.63 \times 10^{-34} \text{ J·s})^2}{(6.67 \times 10^{-11} \text{ N·m}^2/\text{kg}^2)(1.67 \times 10^{-27} \text{ kg})^{8/3}(2 \times 10^{30} \text{ kg})^{1/3}} = 12.3 \text{ km},$$

about the size of a large city. The density would be

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{2 \times 10^{30} \text{ kg}}{\frac{4}{3}\pi(12,300 \text{ m})^3} = 2.6 \times 10^{17} \text{ kg/m}^3,$$

or more than 10^{14} times the density of water. Not surprisingly, this is comparable to the density of an atomic nucleus. The Fermi energy is

$$\begin{aligned}\epsilon_F &= \frac{h^2}{8m_n}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{h^2}{8m_n^{5/3}}\left(\frac{9M}{4\pi^2}\right)^{2/3}\frac{1}{R^2}\\ &= \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(1.67\times10^{-27}\ \text{kg})^{5/3}}\left(\frac{9(2\times10^{30}\ \text{kg})}{4\pi^2}\right)^{2/3}\frac{1}{(12,300\ \text{m})^2}\\ &= 9.1\times10^{-12}\ \text{J} = 5.7\times10^7\ \text{eV},\end{aligned}$$

so the Fermi temperature is

$$T_F = \epsilon_F/k = 6.6\times10^{11}\ \text{K}.$$

This is even higher than for a white dwarf star, so the actual temperature of a neutron star is almost certainly much lower than T_F. Like a white dwarf, a neutron star should become unstable when the neutrons become relativistic, that is, when their average kinetic energy becomes comparable to their rest energy, $mc^2 = 940$ MeV. For a one-solar-mass neutron star, the average kinetic energy is only $0.6\epsilon_F = 34$ MeV, too small by about a factor of 28. But the Fermi energy is proportional to the mass to the 4/3 power, so the critical mass should be larger than the sun's mass by a factor of about $28^{3/4} = 12$. (The experts, however, put the critical mass at only about 3 solar masses, taking into account both density variations and the full relativistic equation of state.)

Problem 7.25. According to equation 7.48, the electronic heat capacity of a mole of copper should be

$$C_V = \frac{\pi^2}{2}\frac{kT}{\epsilon_F}R = \frac{\pi^2}{2}\frac{(8.62\times10^{-5}\ \text{eV/K})(300\ \text{K})}{7.05\ \text{eV}}R = (0.018)R = 0.15\ J/K.$$

For comparison, the heat capacity of lattice vibrations (assuming these are not frozen out) should be roughly $3R = 25$ J/K, 166 times greater. So at room temperature, the electrons contribute less than 1% of the total heat capacity of copper.

Problem 7.26. (Liquid helium-3 as a degenerate Fermi gas.)

(a) The Fermi energy of a "gas" of ^{3}He atoms with the given density is

$$\begin{aligned}\epsilon_F &= \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{(6.63\times10^{-34}\ \text{J·s})^2}{8(3)(1.66\times10^{-27}\ \text{kg})}\left(\frac{3(6.02\times10^{23})}{\pi(37\times10^{-6}\ \text{m}^3)}\right)^{2/3}\\ &= 6.9\times10^{-23}\ J = 4.3\times10^{-4}\ eV.\end{aligned}$$

The Fermi temperature is therefore

$$T_F = \frac{\epsilon_F}{k} = \frac{4.3\times10^{-4}\ \text{eV}}{8.62\times10^{-5}\ \text{J/K}} = 5.0\ \text{K}.$$

That's only a little higher than the boiling point, 3.2 K.

(b) As predicted by equation 7.48, the heat capacity should be

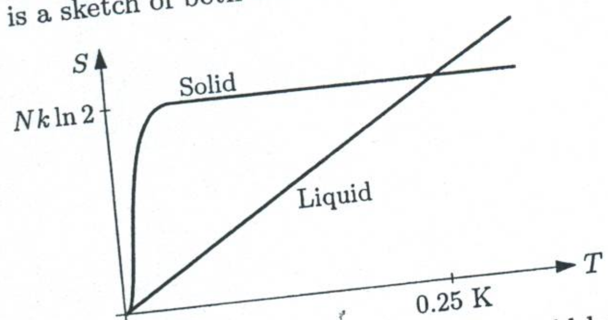
$$\frac{C_V}{NkT} = \frac{\pi^2 k}{2\epsilon_F} = \frac{\pi^2}{2T_F} = 1.0\ \text{K}^{-1}.$$

So although the linear temperature dependence agrees with experiment, the predicted coefficient is too small by almost a factor of 3.

(c) Using the experimental value of the heat capacity coefficient, the entropy of liquid ^{3}He is

$$S = \int_0^T \frac{C_V}{T'}\,dT' = (2.8\ \text{K}^{-1})Nk\int_0^T dT' = (2.8\ \text{K}^{-1})NkT,$$

exactly equal to the heat capacity due to the linear temperature dependence. The entropy of the solid, meanwhile, should be $k\ln 2^N = Nk\ln 2$, since each nucleus has two possible spin orientations. This constant value should apply down to very low (millikelvin or lower) temperatures, when the nuclear spins finally align and the entropy freezes out. Here is a sketch of both entropy functions:



The intersection point where the entropies are equal should be at approximately

$$Nk\ln 2 = (2.8\ \text{K}^{-1})NkT, \quad \text{or} \quad T = \frac{\ln 2}{2.8\ \text{K}^{-1}} = 0.25\ \text{K}.$$

According to the Clausius-Clapeyron relation, the slope of the solid-liquid phase boundary on a graph of P vs. T should be proportional to the entropy difference, $S_{\text{liquid}} - S_{\text{solid}}$. Our analysis therefore predicts that the slope should be positive at temperatures greater than about 0.25 K, and negative at lower temperatures. The experimental phase diagram (Figure 5.13) shows just this behavior, with the transition from positive to negative slope at about 0.3 K, just slightly higher than our prediction. The discrepancy could be because of lattice vibrations giving the solid some additional entropy, and/or the entropy of the liquid no longer being quite linear at relatively high temperatures. At very low temperature, where the entropy of the solid also goes to zero, the phase boundary becomes horizontal.

Problem 7.27. (Heat capacity of a Fermi system with evenly spaced levels.)

(a) Referring to the dot diagrams of Problem 7.16, imagine starting with $q = 0$ and then constructing a state for higher q by displacing one or more solid dots upward. The total number of upward steps taken by the dots must be q, the total number of units

The plus sign gives the physically relevant solution, since the minus sign would give a value of s that actually decreases with increasing t. Squaring this expression then gives

$$\begin{aligned} q &= \frac{\pi^2 t^2}{24}\left(1 + 1 - \frac{24}{\pi^2 t} + 2\sqrt{1 - \frac{24}{\pi^2 t}}\right) = \frac{\pi^2 t^2}{12} - t + \frac{\pi^2 t^2}{12}\sqrt{1 - \frac{24}{\pi^2 t}} \\ &= \frac{\pi^2 t^2}{12} - t + \frac{\pi^2 t^2}{12}\left(1 - \frac{12}{\pi^2 t} + \cdots\right) = \frac{\pi^2 t^2}{6} - 2t + \cdots. \end{aligned}$$

In the second line I've approximated the square root under the assumption that $t \gg 1$, which is true whenever the RH formula applies in the first place. The energy U is just $q\eta$, so the heat capacity is

$$C = \frac{dU}{dT} = k\frac{dq}{dt} \approx k\left(\frac{\pi^2 t}{3} - 2\right) = k\left(\frac{\pi^2}{3}\frac{kT}{\eta} - 2\right).$$

The predicted heat capacity is linear in T, as expected, but offset downward by a constant term. This prediction is plotted as the solid line in the graph above. As you can see, it agrees beautifully with the exact numerical calculation as t becomes large.

Why is the heat capacity of this system independent of N? This may seem like quite a paradox, since heat capacity must be extensive. However, this model system has no explicitly specified volume, so the notion of an extensive vs. intensive quantity is not really meaningful. In real systems, the spacing between energy levels would decrease with increasing volume. So if you like, you can imagine that there is a hidden volume dependence in the constant η. In formula 7.48 for the heat capacity of a Fermi gas in a three-dimensional box, the factor of N really comes from the energy level spacing as well; see equations 7.51 and 7.54.

Problem 7.28. (Two-dimensional Fermi gas.)

(a) In two dimensions, the allowed energy levels are

$$\epsilon = \frac{h^2}{8mL^2}(n_x^2 + n_y^2).$$

At $T = 0$, fermions settle into the lowest unfilled levels, so in two-dimensional n-space, they fill a quarter-circle with radius $n_{\max}$. The Fermi energy is the highest filled level, $\epsilon_F = h^2 n_{\max}^2/8mA$. But the total number of fermions in the system is $N = 2\cdot\pi n_{\max}^2/4$, assuming that the fermions have spin 1/2 and hence two allowed states for each spatial wavefunction. Solving for $n_{\max}^2$ and plugging into the formula for ϵ_F gives

$$\epsilon_F = \frac{h^2}{8mA}\left(\frac{2N}{\pi}\right) = \frac{h^2 N}{4\pi m A}.$$

To compute the total energy, we add up the energies of all filled states and convert the sum to an integral over a quarter-circle in polar coordinates:

$$U = 2\sum_{n_x}\sum_{n_y}\epsilon(\vec{n}) = 2\int_0^{n_{\max}} dn \int_0^{\pi/2} d\phi\, n\,\epsilon(\vec{n}) = \pi\int_0^{n_{\max}} n\,\frac{h^2 n^2}{8mA}\,dn = \frac{\pi h^2 n_{\max}^4}{32mA}.$$

But $n_{max}^2 = 2N/\pi$ so this is just

$$U = \frac{\pi h^2}{32mA}\left(\frac{2N}{\pi}\right)^2 = \frac{h^2N^2}{8\pi mA} = \frac{1}{2}N\epsilon_F.$$

The average energy is just $U/N = \epsilon_F/2$.

(b) To find the density of states, we need to change variables to ϵ in either the integral for the total energy or the integral for the total number of particles. Since the energy integral appears just above, I'll work with it. For the variable change from n to ϵ we need to know that $\epsilon = h^2n^2/8mA$, which implies $d\epsilon = (h^2n/4mA)dn$, or $n\,dn = (4mA/h^2)d\epsilon$. Therefore the energy integral (at $T = 0$) is

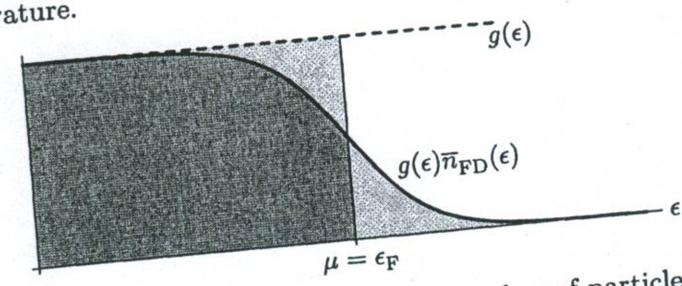
$$U = \int_0^{\epsilon_F} \pi\left(\frac{4mA}{h^2}\right)\epsilon\,d\epsilon \qquad (\text{at } T = 0).$$

We interpret this integral of the sum of all the energies (ϵ) multiplied by the number of states per unit energy ($g(\epsilon)d\epsilon$). Therefore, for this system,

$$g(\epsilon) = \frac{4\pi mA}{h^2} = \frac{N}{\epsilon_F},$$

which is indeed a constant, independent of ϵ.

(c) The illustration below is an adaptation of Figure 7.14 to this two-dimensional system. The solid curve shows the number of particles present, per unit energy; as T increases, the slope of the fall-off becomes shallower. But because $g(\epsilon)$ is constant for this system (and because of the symmetry of the Fermi-Dirac distribution about the point $\epsilon = \mu$), the lightly shaded areas are equal and therefore μ, the point where $\overline{n}_{FD} = 1/2$, remains at its zero-temperature value, ϵ_F. Or almost: At sufficiently high temperatures ($kT \sim \epsilon_F$), the Fermi-Dirac distribution will become significantly less than 1 at *negative* values of ϵ. Since $g(\epsilon) = 0$ at negative ϵ (there are no negative-energy states), the upper lightly-shaded area will then be smaller than the lower one unless μ decreases. At temperatures much greater than ϵ_F/k, the fall-off in the Fermi-Dirac distribution will be so spread out that μ will have to become negative in order to preserve the equality of the two lightly shaded areas. In summary: When $kT \ll \epsilon_F$, μ remains almost exactly equal to ϵ_F. When $kT \gg \epsilon_F$, μ becomes negative and decreases with increasing temperature.



(d) At nonzero temperature, the integral for the total number of particles is

$$N = \int_0^\infty g(\epsilon)\overline{n}_{FD}(\epsilon)\,d\epsilon = g\int_0^\infty \frac{1}{e^{(\epsilon-\mu)/kT}+1}\,d\epsilon,$$

since g is a constant. Changing variables to $x = (\epsilon - \mu)/kT$, this integral becomes

$$N = gkT \int_{-\mu/kT}^{\infty} \frac{1}{e^x + 1}\, dx.$$

The integrand is a composite function involving e^x; if it were multiplied by the derivative of e^x (which is also e^x), we could integrate it easily with another substitution. But we can almost put it into this form by multiplying numerator and denominator by e^{-x}:

$$N = gkT \int_{-\mu/kT}^{\infty} \frac{e^{-x}}{1 + e^{-x}}\, dx.$$

Now just substitute $y = e^{-x}$ and $dy = -e^{-x}dx$:

$$N = -gkT \int_{e^{\mu/kT}}^{0} \frac{1}{1+y}\, dy = -gkT \ln(1+y)\Big|_{e^{\mu/kT}}^{0}$$
$$= -gkT \ln\left(\frac{1}{1+e^{\mu/kT}}\right) = gkT \ln(1 + e^{\mu/kT}).$$

Solving for μ as a function of N then gives

$$\mu = kT \ln(e^{N/gkT} - 1) = kT \ln(e^{\epsilon_F/kT} - 1).$$

When $kT \ll \epsilon_F$, the exponential $e^{\epsilon_F/kT}$ is very large, and the 1 is negligible in comparison, so the right-hand side is approximately $kT \cdot \epsilon_F/kT = \epsilon_F$, as predicted above. When $kT \gg \epsilon_F$, on the other hand, the exponential is only slightly larger than 1, so the argument of the logarithm is less than 1 and therefore the chemical potential is negative as expected.

(e) When $kT \gg \epsilon_F$, the exponential can be expanded in a power series: $1 + \epsilon_F/kT + \cdots$. The 1 cancels, leaving us with

$$\mu \approx kT \ln \frac{\epsilon_F}{kT} = -kT \ln\left(\frac{A}{N}\frac{4\pi mkT}{h^2}\right) = -kT \ln\left(\frac{A}{N}\frac{2}{\ell_Q^2}\right).$$

This is the two-dimensional analogue of equation 6.93 for the chemical potential of an ordinary ("classical") ideal gas, with $Z_{\text{int}} = 2$ because the electron has two internal spin states.

Problem 7.29. The energy integral is

$$U = \int_0^{\infty} \epsilon\, g(\epsilon)\, \overline{n}_{\text{FD}}(\epsilon)\, d\epsilon = g_0 \int_0^{\infty} \epsilon^{3/2}\, \overline{n}_{\text{FD}}(\epsilon)\, d\epsilon.$$

As in equation 7.57, we now integrate by parts:

$$U = \frac{2}{5} g_0 \epsilon^{5/2} \overline{n}_{\text{FD}}(\epsilon)\Big|_0^{\infty} + \frac{2}{5} g_0 \int_0^{\infty} \epsilon^{5/2} \left(-\frac{d\overline{n}_{\text{FD}}}{d\epsilon}\right) d\epsilon.$$

In order to do the rest of the algebra with *Mathematica*, I typed in this expression:

```
energy = (2/5)g0*mu^(5/2) + (Pi^2/4)g0*kT^2*mu^(1/2) -
                                (7*Pi^4/960)g0*kT^4/mu^(3/2)
```

In the next four steps I plugged in the explicit formula for g_0, plugged in the previously calculated series for μ, substituted kT/ϵ_F for t, and expanded everything in a series to fourth order in kT:

```
energy1 = energy /. g0 -> (3/2)(n/eF^(3/2))
energy2 = energy1 /. mu -> muSeries*eF
energy3 = energy2 /. t -> kT/eF
energy4 = Normal[Series[energy3,{kT,0,4}]]
```

The final instruction returned the desired expression for U to fourth order in kT/ϵ_F:

$$U = \frac{3}{5}N\epsilon_F + \frac{\pi^2}{4}N\frac{(kT)^2}{\epsilon_F} - \frac{3\pi^4}{80}N\frac{(kT)^4}{\epsilon_F^3}.$$

Notice that the fourth-order correction to the energy is negative. The corresponding correction to the heat capacity would also be negative, and cubic in temperature, so a plot of C_V vs. T should become concave-down as T becomes comparable to ϵ_F. (See Problem 7.32.)

Problem 7.31. We saw in Problem 7.28 that the density of states of this two-dimensional system is a constant, N/ϵ_F. Therefore the energy integral is

$$U = \frac{N}{\epsilon_F}\int_0^\infty \epsilon\,\overline{n}_{FD}(\epsilon)\,d\epsilon.$$

Unlike the integral for N, this integral cannot be done analytically. So let's integrate by parts as in equation 7.57:

$$U = \frac{N}{\epsilon_F}\frac{\epsilon^2}{2}\overline{n}_{FD}(\epsilon)\Big|_0^\infty - \frac{N}{\epsilon_F}\int_0^\infty \frac{\epsilon^2}{2}\frac{d\overline{n}_{FD}}{d\epsilon}\,d\epsilon.$$

The boundary term vanishes at both limits, leaving us with

$$U = -\frac{N}{2\epsilon_F}\int_0^\infty \epsilon^2\frac{d\overline{n}_{FD}}{d\epsilon}\,d\epsilon = \frac{N}{2\epsilon_F}\int_{-\mu/kT}^\infty \frac{e^x}{(e^x+1)^2}\,\epsilon^2\,dx,$$

where in the last expression I've changed variables to $x = (\epsilon-\mu)/kT$ and inserted expression 7.58 for $d\overline{n}_{FD}/d\epsilon$. So far this expression is exact. But when $kT \ll \epsilon_F$, we can extend the lower limit of the integral down to $-\infty$ as in the three-dimensional case. Since ϵ^2 contains only integer powers of x, no Taylor expansion is necessary; we have simply

$$U = \frac{N}{2\epsilon_F}\int_{-\infty}^\infty \frac{e^x}{(e^x+1)^2}\left[\mu^2 + 2\mu kTx + (kT)^2x^2\right]dx.$$

Evaluating the integrals exactly as on page 284, this becomes

$$U = \frac{N\mu^2}{2\epsilon_F} + \frac{\pi^2}{6}N\frac{(kT)^2}{\epsilon_F} \approx \frac{N\epsilon_F}{2} + \frac{\pi^2}{6}N\frac{(kT)^2}{\epsilon_F}.$$

The heat capacity is therefore

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{\pi^2 N k^2 T}{3\epsilon_F},$$

which is linear in T as expected. Here, however, there are no corrections to the linear behavior that are proportional to higher powers of T; the corrections are instead exponential, just as the deviation of μ from ϵ_F is exponentially suppressed by a factor of $e^{-\epsilon_F/kT}$.

Now consider the high temperature limit, $kT \gg \epsilon_F$. In this limit, as shown in Problem 7.28(e), $\mu \approx kT\ln(\epsilon_F/kT)$, which is negative. Since $e^{(\epsilon-\mu)/kT} \gg 1$ for all ϵ, we can neglect the 1 in the denominator of the Fermi-Dirac distribution and write simply

$$U = \frac{N}{\epsilon_F}\int_0^\infty \epsilon\, e^{-(\epsilon-\mu)/kT}\, d\epsilon = \frac{N}{\epsilon_F}\frac{\epsilon_F}{kT}\int_0^\infty \epsilon\, e^{-\epsilon/kT}\, d\epsilon = \frac{N}{kT}(kT)^2 = NkT,$$

as we would expect for an ordinary ideal gas in two dimensions, according to the equipartition theorem.

Problem 7.32. (Numerical treatment of a Fermi gas.)

(a) Making the substitutions $t = kT/\epsilon_F$, $c = \mu/\epsilon_F$, and $x = \epsilon/\epsilon_F$ in equation 7.53, I obtained the integral

$$1 = \frac{3}{2}\int_0^\infty \frac{\sqrt{x}}{e^{(x-c)/t}+1}\, dx.$$

(Here I've used equation 7.51 for $g(\epsilon)$, and canceled the N's on both sides.) Setting $t = 1$ and $c = 0$, this condition becomes simply

$$1 = \frac{3}{2}\int_0^\infty \frac{\sqrt{x}}{e^{x}+1}\, dx.$$

To evaluate the right-hand side I typed

```
1.5*NIntegrate[Sqrt[x]/(Exp[x]+1),{x,0,Infinity}]
```

into *Mathematica* and got the result 1.017. So μ is not exactly zero when $kT = \epsilon_F$, but it's close. To reduce the value of the integral slightly we would want to make the denominator of the integrand larger, which we can do by making c (or μ) slightly negative.

(b) First I defined a *Mathematica* function to compute the integral for any values of c and t:

```
fermiN[c_,t_] := 1.5*NIntegrate[Sqrt[x]/(Exp[(x-c)/t]+1),{x,0,Infinity}]
```