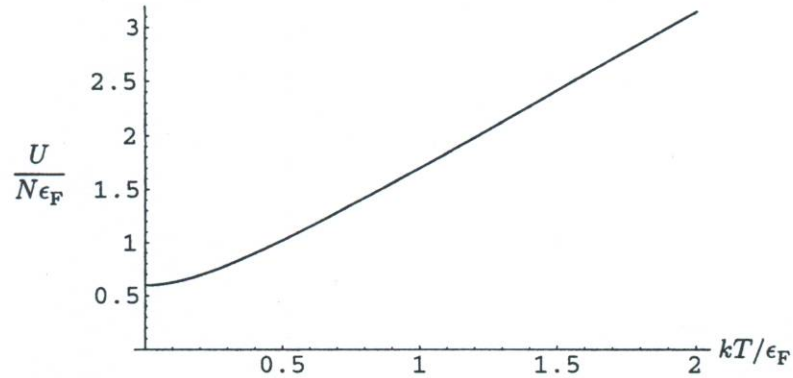
To plot it I gave the instruction

```
Plot[energy[t],{t,.01,2},PlotRange->{All,{0,3.2}}];
```

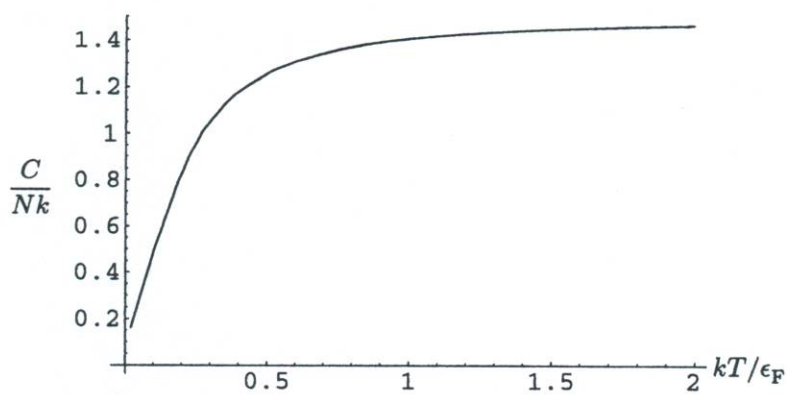
This produced the following graph:



Notice that $U/N\epsilon_F$ goes to $3/5$ at $T = 0$, as expected. To obtain the heat capacity I numerically differentiated the energy function and then plotted the result:

```
heatcap[t_] := (energy[t+.01] - energy[t-.01])/.02
Plot[heatcap[t],{t,.02,1.99}];
```

(Not very elegant, but it does the job.) Note that this "heat capacity" function really calculates C/Nk, since the energy function is really $U/N\epsilon_F$ and t is in units of ϵ_F/k. Here's the plot:



At temperatures much less than ϵ_F/k, the heat capacity is approximately linear in T, as derived in the text. A temperatures much greater than ϵ_F/k, the heat capacity approaches $\frac{3}{2}Nk$, the value for an ordinary "monatomic" ideal gas.

Problem 7.33. (Simple model of a pure semiconductor.)

(a) At nonzero temperature there will be some electrons in the conduction band and some "holes" (unfilled states) in the valence band. But the numbers of conduction electrons and valence holes must be exactly equal, since raising the temperature can't change

the total number of electrons. To calculate the number of electrons within any energy range we would integrate the function $g(\epsilon)\overline{n}_{\rm FD}(\epsilon)$. Now in this simple model, $g(\epsilon)$ is symmetrical about the point $\epsilon = \epsilon_{\rm F}$. Furthermore, as shown in Problem 7.12, the Fermi-Dirac distribution is symmetrical about the point $\epsilon = \mu$, in the sense that the probability of a state at ϵ being occupied is equal to the probability of a state at $\mu - \epsilon$ being unoccupied. In order for the calculated number of conduction electrons to equal the number of valence holes, the produce $g(\epsilon)\overline{n}_{\rm FD}(\epsilon)$ must also have this property, but this will happen only if the symmetry points μ and $\epsilon_{\rm F}$ coincide.

(b) The number of electrons in the conduction band is the integral of the density of states times the Fermi-Dirac distribution:

$$N_c = \int_{\epsilon_c}^{\infty} g(\epsilon)\overline{n}_{\rm FD}(\epsilon)\, d\epsilon = g_0 \int_{\epsilon_c}^{\infty} \sqrt{\epsilon - \epsilon_c}\, \frac{1}{e^{(\epsilon-\epsilon_{\rm F})/kT} + 1}\, d\epsilon.$$

If the width of the gap is much greater than kT, then the exponent in the Fermi-Dirac distribution is large over the whole range of integration and we can neglect the $+1$ in comparison to the exponential. The integral then simplifies considerably; we can evaluate it explicitly by changing variables to $x = (\epsilon - \epsilon_c)/kT$:

$$\begin{aligned} N_c &\approx g_0 \int_{\epsilon_c}^{\infty} \sqrt{\epsilon - \epsilon_c}\, e^{-(\epsilon-\epsilon_{\rm F})/kT}\, d\epsilon \\ &= g_0 \int_{\epsilon_c}^{\infty} \sqrt{\epsilon - \epsilon_c}\, e^{-(\epsilon-\epsilon_c)/kT} e^{-(\epsilon_c-\epsilon_{\rm F})/kT}\, d\epsilon \\ &= g_0 e^{-(\epsilon_c-\epsilon_{\rm F})/kT} (kT)^{3/2} \int_0^{\infty} \sqrt{x}\, e^{-x}\, dx \end{aligned}$$

The integral over x can be done by the methods of Appendix A, or looked up in tables or evaluated by computer; it is simply $\sqrt{\pi}/2$. Plugging in the explicit value of g_0 (from equation 7.51) then gives

$$N_c = \frac{\pi(8m)^{3/2}}{2h^3} V (kT)^{3/2} \frac{\sqrt{\pi}}{2} e^{-(\epsilon_c-\epsilon_{\rm F})/kT} = \frac{2V}{v_Q} e^{-\Delta\epsilon/2kT},$$

where v_Q is the quantum volume defined in equation 7.18 and $\Delta\epsilon$ is the width of the gap, $2 \cdot (\epsilon_c - \epsilon_{\rm F})$.

(c) At room temperature, $kT \approx 0.026$ eV so the exponential factor is

$$e^{-\Delta\epsilon/2kT} = \exp\left(-\frac{1.11 \text{ eV}}{2(0.026 \text{ eV})}\right) = e^{-21.3} = 5.4 \times 10^{-10}.$$

Meanwhile, the quantum volume is

$$v_Q = \left(\frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2\pi(9.11 \times 10^{-31} \text{ kg})(1.38 \times 10^{-23} \text{ J/K})(300 \text{ K})}\right)^{3/2} = 8.0 \times 10^{-26} \text{ m}^3.$$

Therefore the number of conduction electrons per unit volume should be roughly

$$\frac{N_c}{V} = \frac{2(5.4 \times 10^{-10})}{8.0 \times 10^{-26} \text{ m}^3} = 1.3 \times 10^{16} \text{ m}^{-3}.$$

For comparison, the density of conduction electrons in copper is about 8.5×10^{28} per cubic meter, as calculated in Problem 7.19. So copper should conduct electricity roughly a million million times better than (pure) silicon.

(d) A pure semiconductor will conduct much better at higher temperature because there are so many *more* electrons in the conduction band. For example, suppose we raise the temperature of our silicon from 300 K to 350 K. The quantum volume then decreases by a factor of $(350/300)^{3/2} = 1.26$, which isn't very significant. On the other hand, the exponential factor is now

$$e^{-\Delta\epsilon/2kT} = \exp\left(-\frac{1.11 \text{ eV}}{2(8.62 \times 10^5 \text{ eV/K})(350 \text{ K})}\right) = e^{-18.4} = 1.0 \times 10^{-8},$$

greater than its value at 300 K by a factor of 19. The number of conduction electrons is therefore more than 20 times larger than at 300 K. (An ordinary metal, on the other hand, has a fixed number of conduction electrons so its conductivity depends only on how readily these electrons can move around. At high temperatures, collisions occur more frequently, so the conductivity is lower. This effect is also present in silicon, but is swamped by the enormously greater number of conduction electrons.)

(e) To get an insulator (with essentially no conduction electrons), we would want the gap to be significantly wider than the 1.11 eV gap in silicon. Doubling this value should pretty much do it, since this would square the exponential factor computed above, multiplying the number of conduction electrons by another factor of 5×10^{-10} to yield only 7 million per cubic meter. For a 3-eV gap, the exponential factor (at room temperature) is

$$e^{-\Delta\epsilon/2kT} = \exp\left(-\frac{3 \text{ eV}}{2(0.026 \text{ eV})}\right) = e^{-57.7} = 9 \times 10^{-26},$$

so

$$\frac{N_c}{V} = \frac{2(9 \times 10^{-26})}{8.0 \times 10^{-26} \text{ m}^3} = 2 \text{ m}^{-3},$$

negligible indeed.

Problem 7.34. (Semiconductor: asymmetry of the density of states.)

(a) Suppose that $g_{0v} < g_{0c}$. Then, if μ were to remain constant as the temperature increases from zero, the symmetry of the Fermi-Dirac distribution would imply that the number of electrons added to the conduction band is less than the number removed from the valence band. To prevent such nonsense, μ must *decrease* with temperature. Similarly, if $g_{0v} > g_{0c}$, the chemical potential would have to increase with temperature.

(b) The number of electrons in the conduction band is the integral of the density of states times the Fermi-Dirac distribution:

$$N_c = \int_{\epsilon_c}^{\infty} g(\epsilon)\, \overline{n}_{\text{FD}}(\epsilon)\, d\epsilon = g_{0c} \int_{\epsilon_c}^{\infty} \frac{\sqrt{\epsilon - \epsilon_c}}{e^{(\epsilon-\mu)/kT} + 1}\, d\epsilon \approx g_{0c} \int_{\epsilon_c}^{\infty} \sqrt{\epsilon - \epsilon_c}\, e^{-(\epsilon-\mu)/kT}\, d\epsilon,$$

(b) The conduction electrons can be treated as an ordinary ideal gas as long as the 1 in the denominator of the Fermi-Dirac distribution is negligible compared to the exponential function, $e^{(\epsilon-\mu)/kT}$, for all ϵ in the conduction band. This is the same as saying that $\epsilon-\mu \gg kT$ for all ϵ in the conduction band, so a sufficient condition is $\epsilon_c-\mu \gg kT$. In the limit $T \to 0$ this condition definitely holds, since $\epsilon_c-\mu$ goes to a nonzero constant, $I/2$, in this limit. Note also that $\epsilon_c-\mu$ is at least this large at any temperature; furthermore, from the graph in part (a) we see that $\epsilon_c-\mu$ is at least four times greater than kT over the entire range of temperatures plotted. This range includes room temperature, at which $kT = 0.026$ eV so $kT/I = 0.026/0.044 = 0.59$; at this temperature, $\epsilon_c-\mu$ is slightly greater than $3I$, about 5 times greater than kT. Since $e^5 \approx 150$, approximating the Fermi-Dirac distribution by the Boltzmann distribution within the conduction band should be accurate to about the 1% level.

(c) At temperatures around room temperature or lower, the number of valence electrons excited to the conduction band will be quite negligible compared to the number from donor ions, as discussed in part (a). (Actually, the number is even less than in a pure semiconductor, since the chemical potential is considerably higher in this case, implying that occupancies in the valence band are even closer to 1.) At higher temperatures, though, the number of conduction electrons from donor impurities saturates at 10^{17} per cm^3, while the number excited from the valence band continues to rise. At what temperature will this number equal 10^{17} per cm^3? To make a crude estimate, let's go back to the simplified model of a pure semiconductor used in Problem 7.33. There we calculated that at room temperature, pure silicon contains about 10^{10} conduction electrons per cm^3. If we double the temperature to 600 K, the exponential factor increases to

$$e^{-\Delta\epsilon/2kT} = 2.3 \times 10^{-5},$$

about 40,000 times greater than at 300 K. The quantum volume also decreases by a factor of $2^{3/2} = 2.8$, the combined effect is to increase the density of conduction electrons by only a factor of about 10^5. At 900 K, however, the exponential factor increases to 8.1×10^{-4}, greater than at 300 K by a factor of 1.5×10^6. Factor in the decrease in the quantum volume, $(900/300)^{3/2} = 5.2$, and we obtain an increase in the number of conduction electrons by a factor of nearly 10^7, as desired. Of course, all this assumes that the chemical potential is at the midpoint between the valence and conduction bands. In a doped semiconductor, as we've just seen, the chemical potential is considerably higher at relatively low temperatures. But for our phosphorus-in-silicon example, a temperature of 900 K implies $kT/I = 1.76$, at which the chemical potential should drop below ϵ_c by more than 12 times the ionization energy (extrapolating from the graph in part (a)). That puts μ pretty near the center of the band gap, as needed.

Problem 7.36. (Magnetization of a degenerate Fermi gas.)

(a) In the paramagnetic systems studied in Chapters 3 and 6, every elementary dipole was free to flip its spin from up to down—there were no restrictions from states already being occupied. But in a degenerate Fermi gas, most electrons can't flip from one spin alignment to the other, because the state with opposite spin alignment is already occupied by another electron. Only near the Fermi energy are there a significant

number of unoccupied states, so only a small fraction the electrons are free to flip their spins. Therefore, the magnetization should be relatively small, compared to other paramagnets at the same temperature.

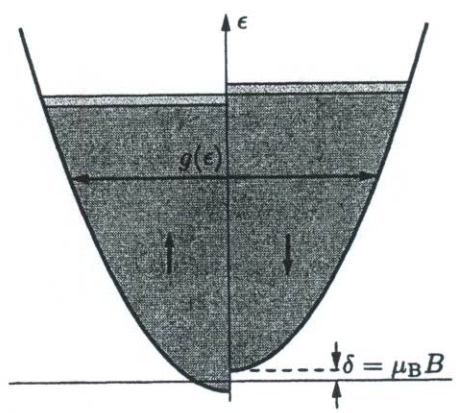
(b) When there is no magnetic field, the density of states is

$$g(\epsilon) = g_0\sqrt{\epsilon}, \qquad \text{where } g_0 = \frac{\pi(8m)^{3/2}}{2h^3} = \frac{3N}{2\epsilon_F^{3/2}}.$$

This formula includes all states with either spin alignment; half of the states at any energy come from each spin alignment. Adding a magnetic field B shifts the energies of half the states (those with positive μ_z, and hence negative z-spin, thanks to the electron's negative charge) down by $\delta \equiv \mu_B B$, and shifts the energies of the other half of the states up by δ. Thus, the density of states is now

$$g(\epsilon) = \frac{g_0}{2}\sqrt{\epsilon+\delta} + \frac{g_0}{2}\sqrt{\epsilon-\delta},$$

where it is understood that the ϵ values in each term are restricted to those that yield real roots. A clever way to interpret this formula graphically is to turn Figure 7.13 on its side, and plot one term to the left and the other to the right:



(c) As the magnetic field is turned on (at $T = 0$), the number of electrons that flip from one spin alignment to the other is equal to the area of either of the lightly shaded regions in the illustration above. Approximating these areas as rectangles, this number is $\delta \cdot \frac{1}{2}g_0\sqrt{\epsilon_F}$. The number of up-dipoles increases by this amount, while the number of down-dipoles decreases by this amount, so the total magnetization is

$$M = \mu_B(N_\uparrow - N_\downarrow) = \mu_B \cdot 2 \cdot \delta \cdot \frac{1}{2}g_0\sqrt{\epsilon_F} = \mu_B^2 B g_0\sqrt{\epsilon_F} = \frac{3N\mu_B^2 B}{2\epsilon_F}.$$

It is interesting to compare this formula to equation 3.35 for an ordinary paramagnet in the regime where Curie's law applies. Aside from the factor of 3/2, the formulas differ only in the replacement of kT by ϵ_F. Since ϵ_F is normally much larger than kT, the present system has a magnetization much less than that of an ordinary paramagnet.

(d) At nonzero T, we can compute the numbers of up- and down-dipoles by integrating the corresponding portion of the density of states, multiplied by the Fermi-Dirac distribution. Therefore the magnetization is

$$M = \mu_B(N_\uparrow - N_\downarrow) = \mu_B\left[\int_{-\delta}^{\infty} \frac{g_0}{2}\sqrt{\epsilon+\delta}\,\overline{n}_{FD}(\epsilon)\,d\epsilon - \int_{\delta}^{\infty} \frac{g_0}{2}\sqrt{\epsilon-\delta}\,\overline{n}_{FD}(\epsilon)\,d\epsilon\right].$$

To evaluate either integral, we can use the Sommerfeld expansion as on pages 283–284 of the text. For instance, the first integral is

$$\int_{-\delta}^{\infty} \sqrt{\epsilon+\delta}\,\overline{n}_{FD}(\epsilon)\,d\epsilon = \frac{2}{3}\int_{-\delta}^{\infty}(\epsilon+\delta)^{3/2}\left(-\frac{d\overline{n}_{FD}}{d\epsilon}\right)d\epsilon \approx \frac{2}{3}\int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}(\epsilon+\delta)^{3/2}\,dx,$$

where in the first step I've integrated by parts and in the second step I've changed variables to $x = (\epsilon-\mu)/kT$ and extended the lower limit of the integral down to infinity. The next step is to expand $(\epsilon+\delta)^{3/2}$ in a power series about $\epsilon = \mu$:

$$(\epsilon+\delta)^{3/2} = (\mu+\delta)^{3/2} + \frac{3}{2}(\epsilon-\mu)(\mu+\delta)^{1/2} + \frac{3}{8}(\epsilon-\mu)^2(\mu+\delta)^{-1/2} + \cdots.$$

Writing $\epsilon-\mu$ as xkT and plugging into the integral above gives (approximately)

$$\begin{aligned}\frac{2}{3}\int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}&\left[(\mu+\delta)^{3/2} + \frac{3}{2}xkT(\mu+\delta)^{1/2} + \frac{3}{8}(xkT)^2(\mu+\delta)^{-1/2}\right]dx \\ &= \frac{2}{3}\left[(\mu+\delta)^{3/2} + \frac{\pi^2}{8}(kT)^2(\mu+\delta)^{-1/2}\right],\end{aligned}$$

where I've evaluated the integrals as in equations 7.62 through 7.64. The other integral is the same except that δ is replaced by $-\delta$, so the magnetization is approximately

$$M = \frac{\mu_B g_0}{2}\cdot\frac{2}{3}\left[(\mu+\delta)^{3/2} - (\mu-\delta)^{3/2} + \frac{\pi^2}{8}(kT)^2(\mu+\delta)^{-1/2} - \frac{\pi^2}{8}(kT)^2(\mu-\delta)^{-1/2}\right].$$

Since δ is much smaller than μ, we can now use the binomial expansion as follows:

$$(\mu\pm\delta)^p = \mu^p\left(1\pm\frac{\delta}{\mu}\right)^p \approx \mu^p\left(1\pm\frac{p\delta}{\mu}\right),$$

where p is either $3/2$ or $-1/2$. Applying this approximation to each term gives

$$\begin{aligned}M &= \frac{\mu_B g_0}{3}\left[\mu^{3/2}\left(1+\frac{3\delta}{2\mu}-1+\frac{3\delta}{2\mu}\right) + \frac{\pi^2}{8}(kT)^2\mu^{-1/2}\left(1-\frac{\delta}{2\mu}-1-\frac{\delta}{2\mu}\right)\right] \\ &= \frac{\mu_B g_0}{3}\left[\mu^{3/2}\cdot\frac{3\delta}{\mu} - \frac{\pi^2}{8}(kT)^2\mu^{-1/2}\cdot\frac{\delta}{\mu}\right] = \mu_B\delta g_0\left[\mu^{1/2} - \frac{\pi^2}{24}\frac{(kT)^2}{\mu^{3/2}}\right].\end{aligned}$$

Now recall that the chemical potential is given, to order $(kT/\epsilon_F)^2$, by equation 7.66 (plus a correction proportional to $(\delta/\epsilon_F)^2$, which we assume to be small in comparison). In the second term of our formula for M, we can simply plug in $\mu = \epsilon_F$, since the

correction would be proportional to $(kT/\epsilon_F)^4$, which we assume to negligible. In the first term, though, we must carefully plug in the corrected formula for μ and use the binomial expansion once again:

$$\begin{aligned} M &= \mu_B \delta g_0 \left[\epsilon_F^{1/2} \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 \right)^{1/2} - \frac{\pi^2}{24} \frac{(kT)^2}{\epsilon_F^{3/2}} \right] \\ &= \mu_B \delta g_0 \left[\epsilon_F^{1/2} \left(1 - \frac{\pi^2}{24} \left(\frac{kT}{\epsilon_F} \right)^2 \right) - \frac{\pi^2}{24} \frac{(kT)^2}{\epsilon_F^{3/2}} \right] \\ &= \mu_B \delta g_0 \epsilon_F^{1/2} \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 \right] = \frac{3N\mu_B^2 B}{2\epsilon_F} \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 \right]. \end{aligned}$$

(In the final step I've used the definitions of δ and g_0.) The prefactor is the same zero-temperature result obtained much more easily in part (c). The correction term shows that as the temperature increases, the magnetization decreases (as one would expect), but only slightly as long as $kT \ll \epsilon_F$.

Problem 7.37. To find the maximum of the Planck function $x^3/(e^x - 1)$, you could just plug in numbers until it becomes clear that $x = 2.82$ gives a larger value than any other x. Or you could use a *Mathematica* instruction such as

```
FindMinimum[-x^3/(Exp[x]-1),{x,3}]
```

which returns the value 2.82144. The sophisticated method is to set the derivative of the function equal to zero:

$$0 = \frac{d}{dx} \left(\frac{x^3}{e^x - 1} \right) = \frac{3x^2(e^x - 1) - x^3 e^x}{(e^x - 1)^2} = \frac{x^2}{(e^x - 1)^2} \left[3(e^x - 1) - xe^x \right].$$

The solutions $x = 0$ and $x = \infty$ give the minima of the function. We want the maximum, which is at the nontrivial solution where

$$3e^x - 3 = xe^x, \qquad \text{or} \qquad e^{-x} = 1 - \frac{x}{3}.$$

But this is a transcendental equation, so we again must resort to numerical methods, such as the *Mathematica* instruction

```
FindRoot[Exp[-x]==1-(x/3),{x,3}]
```

which again returns 2.82144.

Problem 7.38. At $T = 3000$ K, $kT = 0.26$ eV, while at $T = 6000$ K, $kT = 0.52$ eV. To plot the Planck spectrum vs. photon energy at each of these temperatures, I used the *Mathematica* instruction

```
Plot[{e^3/(Exp[e/.26]-1),e^3/(Exp[e/.52]-1)},{e,0,6}]
```