Problem 7.41. (Einstein A and B coefficients.)

(a) The number of atoms in state 1 can increase due to spontaneous decay from state 2, and increase due to stimulated emission from state 2, and decrease due to stimulated absorption up to state 2. The rate of each of these processes is equal to the probability for any one atom undergoing the process (as expressed in terms of the definitions of A, B, and B') times the number of atoms currently in the required initial state. Therefore the total rate is

$$\frac{dN_1}{dt} = AN_2 + B'N_2u(f) - BN_1u(f),$$

where $u(f)$ is evaluated at the required frequency, $f = \epsilon/h$.

(b) In equilibrium, $dN_1/dt = 0$ and N_2/N_1 is given by a simple ratio of Boltzmann factors,

$$\frac{N_2}{N_1} = \frac{e^{-E(s_2)/kT}}{e^{-E(s_1)/kT}} = e^{-\epsilon/kt} = e^{-hf/kT}.$$

The function $u(f)$ is obtained by changing variables from ϵ to $f = \epsilon/h$ in equation 7.83:

$$u(f) = \frac{8\pi h}{c^3}\frac{f^3}{e^{hf/kT} - 1}.$$

(The extra factor of h in the numerator comes from $d\epsilon = h \cdot df$.) Plugging all of these expressions into the result of part (a) and canceling the common factor of N_1, we obtain

$$0 = Ae^{-hf/kT} + (B'e^{-hf/kT} - B)\frac{8\pi h}{c^3}\frac{f^3}{e^{hf/kT} - 1}.$$

Pulling the A term to the left-hand side and multiplying through by $-e^{hf/kT}$ gives

$$A = (Be^{hf/kT} - B') \cdot \frac{8\pi h}{c^3}\frac{f^3}{e^{hf/kT} - 1}.$$

Now this equation must hold for all temperatures T, but the coefficients themselves, being intrinsic properties of the atom, cannot possibly depend on temperature. Therefore, since the left-hand side is independent of temperature, the temperature dependence on the right-hand side must cancel out. The only way this can happen is if $B' = B$. We then have simply

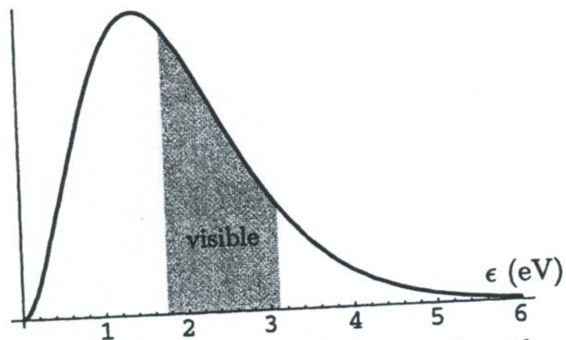
$$A = B(e^{hf/kT} - 1) \cdot \frac{8\pi h}{c^3}\frac{f^3}{e^{hf/kT} - 1} = B \cdot \frac{8\pi h f^3}{c^3},$$

which is Einstein's relation between the rates of spontaneous and stimulated emission.

Problem 7.42. (Electromagnetic radiation in a kiln.)

(a) The total energy of the radiation in a cubic meter of space at 1500 K is

$$U = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}V = \frac{8\pi^5}{15}\frac{[(1.38 \times 10^{-23}\ \text{J/K})(1500\ \text{K})]^4}{[(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(3 \times 10^8\ \text{m/s})]^3}(1\ \text{m}^3) = 0.0038\ \text{J}.$$



To locate the visible portion of the spectrum, note that the red end is at 700 nm which corresponds to a photon energy of $\epsilon = hc/\lambda = 1.77$ eV, while the violet end is at 400 nm which corresponds to a photon energy of 3.1 eV. I've shaded this region in the graph.

(c) Since $kT = 0.50$ eV, the limits of the visible range are at $x = 1.77/0.5 = 3.54$ and $x = 3.1/0.5 = 6.2$. Therefore the fraction of energy in the visible range is

$$\int_{3.54}^{6.2} \frac{x^3}{e^x - 1}\,dx \Big/ \int_0^\infty \frac{x^3}{e^x-1}\,dx = \frac{15}{\pi^4}\int_{3.54}^{6.2}\frac{x^3}{e^x-1}\,dx.$$

I evaluated the integral numerically with the *Mathematica* instruction

```
(15/Pi^4)*NIntegrate[x^3/(Exp[x]-1),{x,3.54,6.2}]
```

which returned the number 0.36831. So about 37% of the sun's energy is within the visible range.

Problem 7.44. (Number of photons in a photon gas.)

(a) To compute the number of photons, we can simply sum the Planck distribution over all "modes," including a factor of 2 to count the two polarization states for each wave shape:

$$N = 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\overline{n}_{\text{Pl}}(\epsilon) = 2\sum_{n_x,n_y,n_z}\frac{1}{e^{hcn/2LkT}-1}.$$

(Except for the absence of a factor of ϵ, this is the same as equation 7.81.) The steps from here on are the same as in the text: Convert the sum to an integral, and carry out the integral in spherical coordinates where the measure includes a factor of n^2 and the angular integrals give a factor of $4\pi/8$. Then change variables to $x = hcn/2LkT$:

$$N = 2\cdot\frac{4\pi}{8}\cdot\int_0^\infty \frac{n^2}{e^{hcn/2LkT}-1}\,dn = \pi\left(\frac{2LkT}{hc}\right)^3\int_0^\infty\frac{x^2}{e^x-1}\,dx$$
$$= 8\pi V\left(\frac{kT}{hc}\right)^3\int_0^\infty\frac{x^2}{e^x-1}\,dx.$$

I evaluated the integral numerically with *Mathematica*:

```
NIntegrate[x^2/(Exp[x]-1),{x,0,Infinity}]
```

It replied with the answer 2.404.

(b) Combining this result with equation 7.89, I find for the entropy per photon

$$\frac{S}{N} = \frac{(32\pi^5/45)V(kT/hc)^3 k}{2.404 \cdot 8\pi V(kT/hc)^3} = \frac{32\pi^5/45}{2.404 \cdot 8\pi} k = 3.60k.$$

So in fundamental units, the entropy per photon is 3.6.

(c) At room temperature,

$$\frac{N}{V} = 2.404 \cdot 8\pi \left(\frac{kT}{hc}\right)^3 = 60.4 \left(\frac{(8.62 \times 10^{-5}\ \text{eV/K})(300\ \text{K})}{(4.14 \times 10^{-15}\ \text{eV} \cdot \text{s})(3 \times 10^8\ \text{m/s})}\right)^3 = 5.5 \times 10^{14}\ \text{m}^{-3}.$$

(This may seem large, but it's tiny compared to the number of air molecules per cubic meter under atmospheric conditions.) At 1500 K the temperature is 5 times as high so we can simply multiply by 5^3 to obtain $N/V = 6.8 \times 10^{16}\ \text{m}^{-3}$. And for 2.73 K we can multiply by $(2.73/300)^3$ to obtain $4.1 \times 10^8\ \text{m}^{-3}$. That's slightly under half a billion photons per cubic meter filling the entire observable universe—huge compared to the average density of ordinary matter.

Problem 7.45. To evaluate $(\partial U/\partial V)_{S,N}$, we need a formula for U in terms of V and S; we need to eliminate T, and holding S fixed is the same as holding N fixed. Let me abbreviate $\alpha = 8\pi^5 k^4/15(hc)^3$, so

$$U = \alpha V T^4 \qquad \text{and} \qquad S = \frac{4}{3}\alpha V T^3.$$

Solving the second equation for T and plugging into the first gives

$$U = \alpha V \left(\frac{3S}{4\alpha V}\right)^{4/3} = \left(\frac{3S}{4}\right)^{4/3} (\alpha V)^{-1/3}.$$

Now we can compute

$$P = -\left(\frac{\partial U}{\partial V}\right)_S = \frac{1}{3}\left(\frac{3S}{4}\right)^{4/3} \alpha^{-1/3} V^{-4/3} = \frac{1}{3}\frac{U}{V} = \frac{1}{3}\alpha T^4.$$

At 1500 K, the energy density is 0.0038 J/m^3, as computed in Problem 7.42(a). The pressure is therefore 1/3 of this, or 0.0013 Pa. For comparison, the pressure of the air inside an ordinary kiln is the same as outside, approximately 1 atm (if the kiln is at sea level) or 10^5 Pa, greater by a factor of almost 10^8. On the other hand, at the center of the sun the temperature is greater by a factor of 10^4, so the radiation pressure should be greater by a factor of $(10^4)^4$, that is, about 1.3×10^{13} Pa. For comparison, the ionized hydrogen would have a pressure of roughly

$$P = \frac{nRT}{V} = 2(10^3\ \text{mol/kg})(10^5\ \text{kg/m}^3)(8.3\ \text{J/mol·K})(1.5 \times 10^7\ \text{K}) = 2.5 \times 10^{16}\ \text{Pa},$$

where the factor of 2 accounts for the two particles (electron and proton) per ionized atom. This is still greater than the radiation pressure, but "only" by a factor of about 2000. My understanding is that there are some stars in which the radiation pressure is actually larger than the gas pressure.

Problem 7.46. (Free energy of a photon gas.)

(a) From equations 7.86 and 7.89, we have

$$F = U - TS = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}V - T\cdot\frac{32\pi^5}{45}V\left(\frac{kT}{hc}\right)^3 k$$
$$= \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}V\left(1-\frac{4}{3}\right) = -\frac{8\pi^5}{45}\frac{(kT)^4}{(hc)^3}V = -\frac{1}{3}U.$$

(b) Differentiating this result with respect to T gives

$$\left(\frac{\partial F}{\partial T}\right)_V = -\frac{32\pi^5}{45}\frac{k^4T^3}{(hc)^3}V,$$

which is indeed equal to $-S$, by equation 7.89.

(c) By equation 5.22 and the result of part (a),

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{8\pi^5}{45}\frac{(kT)^4}{(hc)^3} = \frac{1}{3}\frac{U}{V},$$

in agreement with the result of the previous problem.

(d) For any particular mode with energy ϵ, the partition function is $Z = (1-e^{-\epsilon/kT})^{-1}$, as calculated in equation 7.70. Therefore the free energy of this mode is $F = -kT\ln Z = kT\ln(1-e^{-\epsilon/kT})$. To get the total free energy, we sum this expression over all modes, as in equation 7.81:

$$F = 2\sum_{n_x,n_y,n_z} kT\ln(1-e^{-\epsilon/kT}) = 2kT\cdot\frac{\pi}{2}\int_0^\infty n^2\ln(1-e^{-\epsilon/kT})\,dn.$$

In the last expression I've converted the sum to an integral in spherical coordinates over the first octant of n-space, and carried out the angular integrals to obtain $\pi/2$, the area of an eighth of a unit sphere. Changing variables to $x = \epsilon/kT = hcn/2LkT$ then gives

$$F = \pi kT\left(\frac{2LkT}{hc}\right)^3\int_0^\infty x^2\ln(1-e^{-x})\,dx = 8\pi V\frac{(kT)^4}{(hc)^3}\int_0^\infty x^2\ln(1-e^{-x})\,dx.$$

To put this integral into a more familiar form, integrate by parts; that is, integrate the x^2 to obtain $x^3/3$, and differentiate the logarithm:

$$F = 8\pi V\frac{(kT)^4}{(hc)^3}\left[\frac{x^3}{3}\ln(1-e^{-x})\Big|_0^\infty - \int_0^\infty\frac{x^3}{3}\frac{e^{-x}}{1-e^{-x}}\,dx\right]$$

The boundary term vanishes at both limits, so we're left with

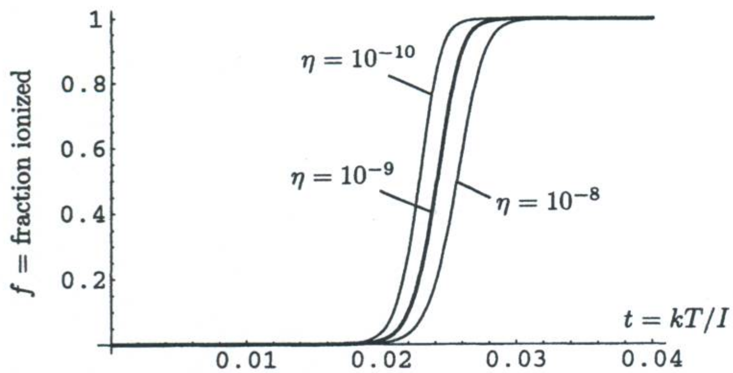
$$F = -\frac{8\pi V}{3}\int_0^\infty\frac{x^3}{e^x-1}\,dx = -\frac{1}{3}U,$$

by comparison with equation 7.85. This is the same result obtained in part (a).

where I've plugged in the numerical values $I = 13.6$ eV and $mc^2 = 511,000$ eV. To plot f as a function of t for $\eta = 10^{-9}$, I used the following *Mathematica* code:

```
eta = 10^-9;
alpha = eta*(5.27*10^-7)*(t^1.5)*Exp[1/t];
Plot[(Sqrt[1+4*alpha]-1)/(2*alpha),{t,0,.04},PlotRange->All]
```

(The upper limit on the t range corresponds to $T = (0.04)I/k = 6300$ K.) I also repeated the calculation for $\eta = 10^{-8}$ and $\eta = 10^{-10}$. The results are shown in the plot below:



First note that the atoms go from completely ionized to completely un-ionized over a very narrow range of temperatures, and that the temperature of the transition is relatively insensitive to the value of η. A larger value of η means that the atoms are closer together at a given temperature, so there is a greater tendency for them *not* to be ionized. Furthermore, the transition temperature is considerably lower than I/k, basically due to the greater entropy of the ionized state. The temperature at which essentially all the hydrogen is un-ionized is roughly $(0.02)I/k \approx 3000$ K. This is roughly the temperature at which the universe first became transparent. However, this calculation assumes thermal equilibrium, and it turns out that the universe was expanding and cooling too quickly for neutral hydrogen to form at the rate required for equilibrium; see Peebles (1993). More sophisticated calculations show that there was still a small amount of ionized plasma left at temperatures below 3000 K, but not enough to keep the universe opaque to photons. [Note: The formula for f predicts that when $t \gg 1$, f decreases and goes to zero as $t \to \infty$. This is because we're assuming that the density of hydrogen is proportional to the density of photons, which is extremely high at high temperatures, and the Saha equation says that the atoms prefer not to be ionized at such high density. However, the t value at which the ionization fraction would drop significantly below 1 is of order 10^{10}, that is, $kT \sim 10^{11}$ eV. This temperature is so high that the conditions would be extremely different from those assumed in this problem.]

Problem 7.48. (The cosmic neutrino background.)

(a) According to equation 5.102, the condition for equilibrium is the same as the reaction equation, but with the name of each species replaced by its chemical potential. So, for the reaction $\nu + \bar{\nu} \leftrightarrow 2\gamma$, the equilibrium condition would be

$$\mu_\nu + \mu_{\bar{\nu}} = 2\mu_\gamma.$$

But the chemical potential for the photons is zero as discussed on page 290, while the chemical potentials of the neutrinos and antineutrinos are equal to each other if they are equally abundant. Therefore we must have $\mu_\nu = \mu_{\bar{\nu}} = 0$.

(b) The probability of any single-particle state being occupied by a neutrino should be given by the Fermi-Dirac distribution, with $\mu = 0$ as shown in part (a). To find the total energy of all such particles in a box, we multiply by $\epsilon = hcn/2L$ (assuming massless neutrinos) and sum over states just as for photons:

$$U = 3 \cdot 2 \cdot \sum_{n_x} \sum_{n_y} \sum_{n_z} \frac{\epsilon}{e^{hcn/2LkT} + 1}.$$

Here the factor of 3 counts the three neutrino species, and the factor of 2 counts the neutrinos and antineutrinos. Now convert the triple sum to an integral in spherical coordinates, evaluate the angular integrals to obtain $4\pi/8$, and change variables to $x = hcn/2LkT$ as always:

$$U = 6 \cdot \frac{4\pi}{8} \int_0^\infty \frac{(hcn/2L)n^2}{e^{hcn/2LkT} + 1}\, dn = 3\pi \Big(\frac{2L}{hc}\Big)^3 (kT)^4 \int_0^\infty \frac{x^3}{e^x + 1}\, dx$$

As shown in Appendix B (equation B.36), this integral (with a + in the denominator) is equal to 7/8 times the integral we did for the photon gas (which had a − in the denominator). Therefore,

$$\frac{U}{V} = \frac{24\pi (kT)^4}{(hc)^3} \cdot \frac{7}{8} \cdot \frac{\pi^4}{15} = \frac{7\pi^5 (kT)^4}{5(hc)^3}.$$

This is exactly 21/8 times the energy density of photons at the same temperature.

(c) The number of neutrinos is given by the same calculation, but without the extra factor of ϵ in the numerator:

$$N = 3\pi \int_0^\infty \frac{n^2}{e^{hcn/2LkT} + 1}\, dn = 3\pi \Big(\frac{2LkT}{hc}\Big)^3 \int_0^\infty \frac{x^2}{e^x + 1}\, dx.$$

This integral is 3/4 times the corresponding integral for photons (evaluated in Problem 7.44), so

$$\frac{N}{V} = 24\pi \Big(\frac{kT}{hc}\Big)^3 \cdot \frac{3}{4} \cdot 2.404 = (135.9) \Big(\frac{kT}{hc}\Big)^3.$$

At $T = 1.95$ K, this evaluates to

$$\frac{N}{V} = (135.9) \bigg(\frac{(8.62 \times 10^{-5}\ \text{eV/K})(1.95\ \text{K})}{(4.14 \times 10^{-15}\ \text{eV}\cdot\text{s})(3 \times 10^8\ \text{m/s})}\bigg)^3 = 3.4 \times 10^8\ \text{m}^{-3},$$

just slightly less than the present density of photons in the universe.

(d) For a single species of neutrino and antineutrino, the present number density would be 1/3 of the number just calculated, or 1.1×10^8 per cubic meter. The average density of ordinary matter in the universe, on the other hand, is only about one proton per cubic meter, or, multiplying by c^2 to get the energy equivalent, about 1 GeV or 10^9 eV per cubic meter. To equal this energy density, the neutrinos would need an energy (mc^2) of only about 10 eV each, since there are roughly 10^8 of them. This is comparable to the present experimental upper limit on the mass of the electron neutrino, but much less than the experimental limits on the masses of the other two species. (By contrast, the lightest particle that is known to be massive is the electron, with $mc^2 = 5 \times 10^5$ eV.)