

(c) On a per-kilogram basis, my radiation rate is

$$\frac{1000 \text{ W}}{75 \text{ kg}} = 14 \text{ W/kg},$$

whereas the sun's is

$$\frac{3.9 \times 10^{26} \text{ W}}{2 \times 10^{30} \text{ kg}} = 0.0002 \text{ W/kg},$$

about 70,000 times less (and 7000 times less than my rate of fuel consumption). How is this possible? Although the sun is bright, it is also very massive. And although it generates energy by nuclear fusion, the reactions in its core actually proceed extremely slowly—giving it a ten-billion-year lifetime. I, on the other hand, have to replenish my (chemical) fuel supply on a daily basis.

Problem 7.53. (Hawking radiation from black holes.)

(a) We calculated in Problem 3.7 that the temperature of a one-solar-mass black hole is 6×10^{-8} K. For a blackbody at this temperature, the peak in the radiation spectrum (plotted as a function of photon energy) would be at $\epsilon = (2.82)kT = 1.5 \times 10^{-11}$ eV. This corresponds to a wavelength of $\lambda = hc/\epsilon = 84$ km. More generally, the peak would be at a wavelength of

$$\lambda = \frac{hc}{(2.82)kT} = \frac{hc}{2.82} \cdot \frac{16\pi^2 GM}{hc^3} = (28.0) \frac{2GM}{c^2}.$$

The quantity $2GM/c^2$ is just the “radius” of the black hole, that is, the quantity that you could plug into the formula $4\pi r^2$ to obtain the surface area. Thus, for any black hole, the typical wavelength emitted is about 28 times the “radius,” or 14 times the “diameter.”

(b) The total power radiated should be given by Stefan's law:

$$\text{power} = \sigma AT^4 = \left(\frac{2\pi^5 k^4}{15h^3 c^2} \right) \left(\frac{16\pi^2 G^2 M^2}{c^4} \right) \left(\frac{hc^3}{16\pi^2 kGM} \right)^4 = \frac{hc^6}{(30,720)\pi^6 G^2 M^2}.$$

For the sun's mass (2×10^{30} kg), this expression evaluates to 9×10^{-31} watts, or 6×10^{-12} eV/s. Since the typical photon radiated has an energy of 1.5×10^{-11} eV, this means that such a black hole would emit a (very feeble) photon only about once every two or three seconds.

(c) The power radiated is the same as the rate at which the black hole's energy (Mc^2) decreases, so the rate of decrease is given by the differential equation

$$\frac{d(Mc^2)}{dt} = -\sigma AT^4 = -\frac{hc^6}{(30,720)\pi^6 G^2 M^2}.$$

That is,

$$\frac{dM}{dt} = -\frac{H}{M^2}, \quad \text{where} \quad H = \frac{hc^6}{(30,720)\pi^6 G^2} = 4.0 \times 10^{15} \text{ kg}^3/\text{s}.$$

(We could refer to H as Hawking's constant.) This is a separable differential equation, which we can integrate to obtain the lifetime τ of the black hole:

$$\int_{M_i}^0 M^2 dM = -H \int_0^\tau dt \quad \Rightarrow \quad -\frac{M_i^3}{3} = -H\tau,$$

that is, $\tau = M_i^3/3H$.

(d) For $M_i = 2 \times 10^{30}$ kg, the lifetime should be

$$\tau = \frac{(2 \times 10^{30} \text{ kg})^3}{3(4.0 \times 10^{15} \text{ kg}^3/\text{s})} = 7 \times 10^{74} \text{ s}.$$

That's 2×10^{67} years, or more than 10^{57} times the age of the known universe. Black holes that form by stellar collapse should have initial masses at least this large, so there's no hope of observing such black holes disappearing any time soon.

(e) The age of the known universe is about 15 billion years or 5×10^{17} seconds. The initial mass of a black hole with this lifetime would be

$$M_i = (3H\tau)^{1/3} = [3(4 \times 10^{15} \text{ kg}^3/\text{s})(5 \times 10^{17} \text{ s})]^{1/3} = 1.8 \times 10^{11} \text{ kg},$$

smaller than the sun's mass by a factor of about 10^{19} . The "radius" of such a black hole would be

$$\frac{2GM}{c^2} = 2.6 \times 10^{-16} \text{ m},$$

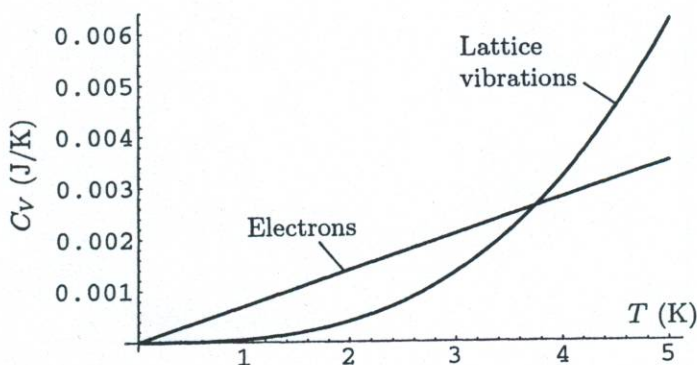
and therefore the radiation it emits (initially) would peak at a wavelength of about 28 times this, or 7 femtometers. That's comparable to the size of an atomic nucleus. A photon with this wavelength has an energy of $\epsilon = hc/\lambda = 170$ MeV. That's a very hard gamma ray, a hundred times more energetic than gamma rays emitted in nuclear reactions, though not as energetic as those produced at today's particle accelerators. As the black hole evaporates and loses mass, its temperature increases and the gamma rays emitted become even more energetic. However, a black hole that can emit MeV gamma rays can probably also emit electron-positron pairs and perhaps other species of massive particles. This would increase its rate of evaporation and decrease its lifetime. Therefore, to have lasted the age of the universe, a black hole probably would have needed an initial mass somewhat greater than I've calculated.

Problem 7.54. (Stellar surface temperatures and sizes.) Stefan's law, in conventional units, reads $L = \sigma AT^4 = 4\pi\sigma R^2 T^4$, where L is the star's luminosity and R is its radius. For convenience, though, we could measure L , R , and T in units of the sun's values. In these units, the constant $4\pi\sigma$ must be equal to 1, because the sun's temperature and radius (both 1) must yield the sun's luminosity (1). Solving the equation for R then gives simply $R = \sqrt{L/T^4}$. Note also that the energy at which a blackbody spectrum peaks is directly proportional to the temperature ($\epsilon = (2.82)kT$), so the ratio of a star's temperature to that of the sun is the same as the ratio of the peak photon energies. As calculated on page 305, the sun's spectrum peaks at a photon energy of 1.41 eV.

where α is an abbreviation for $12\pi^4 Nk/5T_D^3$, the slope of the graph plotted in Figure 7.28. From the data for copper plotted in the figure, this slope appears to be roughly $(0.9 \text{ mJ/K}^2)/(18 \text{ K}^2) = 5 \times 10^{-5} \text{ J/K}^4$, while γ , the intercept, appears to be roughly 0.7 mJ/K^2 . Therefore the temperature at which the two contributions are equal should be

$$T = \sqrt{\frac{7 \times 10^{-4} \text{ J/K}^2}{5 \times 10^{-5} \text{ J/K}^4}} = \sqrt{14 \text{ K}^2} = 3.7 \text{ K}.$$

At this temperature, each of the contributions to the heat capacity is equal to $\gamma T = (7 \times 10^{-4} \text{ J/K}^2)(3.7 \text{ K}) = 0.0026 \text{ J/K}$. Here, then, is a plot of the two contributions separately:



Problem 7.61. If we repeat the derivation on pages 308 through 311 for the case of a liquid, the only thing that changes is the number of polarization states for each triplet (n_x, n_y, n_z) : now there is only one polarization rather than three. This change has no effect on volume of n -space that is summed over, and therefore has no effect on the formula $n_{\text{max}} = (6N/\pi)^{1/3}$ or on equation 7.111 for the Debye temperature:

$$T_D = \frac{hc_s}{2k} \left(\frac{6N}{\pi V} \right)^{1/3}.$$

To evaluate this expression we need to know the ratio N/V . Let's take N to be Avogadro's number, so that the mass of the sample is 4 g. At a density of 0.145 g/cm^3 , this value implies a volume of 27.6 cm^3 or $2.76 \times 10^{-5} \text{ m}^3$. The predicted Debye temperature is therefore

$$T_D = \frac{(6.63 \times 10^{-32} \text{ J}\cdot\text{s})(238 \text{ m/s})}{2(1.38 \times 10^{-23} \text{ J/K})} \left(\frac{6(6.02 \times 10^{23})}{\pi(2.76 \times 10^{-5} \text{ m}^3)} \right)^{1/3} = 19.8 \text{ K}.$$

What does change in the derivation in the text is the numerical factor multiplying the energy (and the heat capacity): The factor of 3 in equation 7.106 disappears, so each expression for U or C_V from there on should be divided by 3 for the case of a liquid. The heat capacity in the low-temperature limit is therefore $1/3$ times the formula in equation 7.115:

$$\frac{C_V}{Nk} = \frac{4\pi^4}{5} \left(\frac{T}{T_D} \right)^3.$$

The cube root of $5/4\pi^4$ is 0.234, so this is the same as

$$\frac{C_V}{Nk} = \left(\frac{T}{(0.234)T_D} \right)^3 = \left(\frac{T}{4.64 \text{ K}} \right)^3,$$

in almost perfect agreement with the measured behavior.

Problem 7.62. When $T \gg T_D$, the x values integrated over in equation 7.112 are all much less than 1, so we can expand the exponential in a power series:

$$\begin{aligned} \frac{x^3}{e^x - 1} &\approx \frac{x^3}{(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3) - 1} = \frac{x^3}{x + \frac{1}{2}x^2 + \frac{1}{6}x^3} = x^2(1 + \frac{1}{2}x + \frac{1}{6}x^2)^{-1} \\ &\approx x^2 \left[1 - (\frac{1}{2}x + \frac{1}{6}x^2) + \frac{1}{2}(-1)(-2)(\frac{1}{2}x + \frac{1}{6}x^2)^2 \right] \\ &\approx x^2 \left[1 - \frac{1}{2}x - \frac{1}{6}x^2 + \frac{1}{4}x^2 \right] = x^2 - \frac{1}{2}x^3 + \frac{1}{12}x^4. \end{aligned}$$

Integrating this expression from 0 to T_D/T , we obtain

$$\begin{aligned} U &\approx \frac{9NkT^4}{T_D^3} \left[\frac{1}{3} \left(\frac{T_D}{T} \right)^3 - \frac{1}{8} \left(\frac{T_D}{T} \right)^4 + \frac{1}{60} \left(\frac{T_D}{T} \right)^5 \right] \\ &= 9NkT_D \left[\frac{1}{3} \left(\frac{T}{T_D} \right) - \frac{1}{8} + \frac{1}{60} \left(\frac{T_D}{T} \right) \right]. \end{aligned}$$

To obtain the heat capacity, differentiate:

$$C_V = \frac{\partial U}{\partial T} = 9NkT_D \left[\frac{1}{3T_D} - 0 - \frac{T_D}{60T^2} \right] = 3Nk \left[1 - \frac{1}{20} \left(\frac{T_D}{T} \right)^2 \right].$$

Assuming that this formula is reasonably accurate down to $T = T_D$, it predicts that the heat capacity is 5% below its asymptotic value at $T = T_D$, and 1.25% below its asymptotic value at $T = 2T_D$.

Problem 7.63. For a two-dimensional material, the average energy in each wave mode will still be given by the Planck distribution, but to compute the total thermal energy we carry out only a double sum over modes:

$$U = \sum_{n_x} \sum_{n_y} \frac{\epsilon}{e^{\epsilon/kT} - 1},$$

where $\epsilon = hf = hc_s/\lambda = hc_s n/2L = (hc_s/2L) \sqrt{n_x^2 + n_y^2}$. The numerical factor in front of the sum is 1, assuming that each mode has only one possible polarization (compare equation 7.106). If the material is in the shape of a square and there are N atoms, then each sum goes from 1 to \sqrt{N} , the number of modes along each direction. In other words, the sum is over a square region in n -space with area N (see the illustration on the following page). Assuming that N is large, we can replace the sum by a double integral over the same region:

$$U = \int_0^{\sqrt{N}} dn_x \int_0^{\sqrt{N}} dn_y \frac{\epsilon}{e^{\epsilon/kT} - 1}.$$

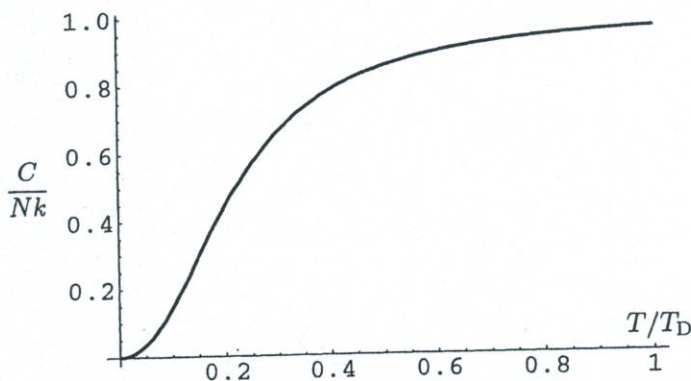
To obtain the heat capacity at intermediate temperatures, it's easiest to differentiate the energy integral before changing variables to x :

$$\begin{aligned} C_A &= \frac{\partial U}{\partial T} = \frac{\pi}{2} \int_0^{n_{\max}} \frac{\partial}{\partial T} \left(\frac{n \epsilon}{e^{\epsilon/kT} - 1} \right) dn = \frac{\pi}{2} \int_0^{n_{\max}} \frac{n \epsilon (\epsilon/kT^2) e^{\epsilon/kT}}{(e^{\epsilon/kT} - 1)^2} dn \\ &= \frac{\pi}{2} \cdot k \cdot \left(\frac{2LkT}{hc_s} \right)^2 \int_0^{x_{\max}} \frac{x^3 e^x}{(e^x - 1)^2} dx = \frac{\pi}{2} \cdot k \cdot \frac{4N}{\pi} \left(\frac{T}{T_D} \right)^2 \int_0^{x_{\max}} \frac{x^3 e^x}{(e^x - 1)^2} dx \\ &= \frac{2NkT^2}{T_D^2} \int_0^{x_{\max}} \frac{x^3 e^x}{(e^x - 1)^2} dx. \end{aligned}$$

To plot C/Nk vs. T/T_D , I gave *Mathematica* the instruction

```
Plot[2*t^2*NIntegrate[x^3*Exp[x]/(Exp[x]-1)^2,{x,0,1/t}},{t,0,1}]
```

and it produced the following graph:



Although this graph looks similar to that for a three-dimensional solid (Figure 7.29), here the low-temperature behavior is quadratic (rising more suddenly) instead of cubic.

Problem 7.64. (Spin waves in a ferromagnet.)

- (a) The total number of magnons at temperature T should be given by the Planck distribution, summed over all modes:

$$N_m = \sum_{n_x, n_y, n_z} \frac{1}{e^{\epsilon/kT} - 1},$$

where $\epsilon = p^2/2m^*$, $p = h/\lambda = hn/2L$, and $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$. If we convert the sum to an integral in spherical coordinates, the angular integrals give a factor of $\pi/2$ (as always), leaving us with

$$N_m = \frac{\pi}{2} \int_0^{\infty} \frac{n^2}{e^{\epsilon/kT} - 1} dn,$$

where I've used ∞ as the upper limit because this whole picture applies only at relatively low temperatures. Now change variables to $x = \epsilon/kT$:

$$x = \frac{p^2}{2m^*kT} = \frac{h^2 n^2}{8m^*L^2kT}, \quad n = \sqrt{\frac{8m^*L^2kTx}{h^2}}, \quad dn = \sqrt{\frac{2m^*L^2kT}{h^2}} \frac{1}{\sqrt{x}} dx.$$

This variable change puts the expression for N_m into the form

$$N_m = \frac{\pi}{2} \left(\frac{8m^*L^2kT}{h^2} \right) \sqrt{\frac{2m^*L^2kT}{h^2}} \int_0^\infty \frac{\sqrt{x}}{e^x - 1} dx = 2\pi V \left(\frac{2m^*kT}{h^2} \right)^{3/2} \int_0^\infty \frac{\sqrt{x}}{e^x - 1} dx.$$

According to *Mathematica*, the integral is equal to 2.315.

- (b) If the total magnetization at $T = 0$ is $2\mu_B N$, and each magnon reduces this value by $2\mu_B$, then the fractional reduction in magnetization is

$$\frac{2\mu_B N_m}{2\mu_B N} = \frac{N_m}{N} = 2\pi(2.315) \frac{V}{N} \left(\frac{2m^*kT}{h^2} \right)^{3/2} = \left(\frac{T}{T_0} \right)^{3/2},$$

where

$$T_0 = \frac{h^2}{2m^*k} \left(\frac{N}{V} \right)^{2/3} \frac{1}{(2 \cdot \pi \cdot 2.315)^{2/3}} = \frac{(0.0839)h^2}{m^*k} \left(\frac{N}{V} \right)^{2/3}.$$

For iron, we're given $m^* = 1.24 \times 10^{-29}$ kg. The ratio N/V can be calculated from the density and the atomic mass, or we can look up V/N_{avo} on page 404. So for iron, we can predict

$$T_0 = \frac{(0.0839)(6.63 \times 10^{-34} \text{ J}\cdot\text{s})^2}{(1.24 \times 10^{-29} \text{ kg})(1.38 \times 10^{-23} \text{ J/K})} \left(\frac{6.02 \times 10^{23}}{7.11 \times 10^{-6} \text{ m}^3} \right)^{2/3} = 4150 \text{ K}.$$

So the temperature has to be pretty high before the magnetization decreases by a substantial fraction.

- (c) To calculate the heat capacity, we should first calculate the energy:

$$U = \sum_{n_x, n_y, n_z} \frac{\epsilon}{e^{\epsilon/kT} - 1} = \frac{\pi}{2} \int_0^\infty \frac{\epsilon n^2}{e^{\epsilon/kT} - 1} dn = 2\pi V \left(\frac{2m^*kT}{h^2} \right)^{3/2} (kT) \int_0^\infty \frac{x^{3/2}}{e^x - 1} dx.$$

Mathematica says that this integral equals 1.783, so

$$U = 2\pi(1.783)Vk \left(\frac{2m^*k}{h^2} \right)^{3/2} T^{5/2} = (31.69)Vk \left(\frac{m^*k}{h^2} \right)^{3/2} T^{5/2}.$$

Differentiating with respect to T gives the heat capacity:

$$\frac{C_V}{Nk} = \frac{1}{Nk} \frac{\partial U}{\partial T} = (31.69) \frac{5}{2} \frac{V}{N} \left(\frac{m^*k}{h^2} \right)^{3/2} = \left(\frac{T}{T_1} \right)^{3/2},$$

where

$$T_1 = \frac{h^2}{m^*k} \left(\frac{N}{V} \right)^{2/3} \left(\frac{2}{5(31.69)} \right)^{2/3} = \left(\frac{0.0542}{0.0839} \right) T_0 = (0.646)T_0.$$

For iron, therefore, $T_1 = 2680$ K, which implies that the magnon contribution to the heat capacity is quite small (compared to Nk) at room temperature and below. However, at sufficiently low temperatures, the magnon contribution will be greater

than the phonon contribution, which is proportional to T^3 (see equation 7.115). The temperature at which these two contributions are equal is given by

$$\left(\frac{T}{T_1}\right)^{3/2} = \frac{12\pi^4}{5} \left(\frac{T}{T_D}\right)^3, \quad \text{or} \quad T = \left(\frac{5}{12\pi^4}\right)^{2/3} \frac{T_D^2}{T_1} = (0.0264) \frac{(470 \text{ K})^2}{2680 \text{ K}} = 2.17 \text{ K}.$$

Thus, at temperatures of a few kelvin, the magnon contribution to the heat capacity should be measurable.

- (d) For a similar system in two dimensions, the number of magnons at temperature T should be

$$N_m = \sum_{n_x, n_y} \frac{1}{e^{\epsilon/kT} - 1} = \frac{\pi}{2} \int_0^\infty \frac{n}{e^{\epsilon/kT} - 1} dn,$$

where I've converted the sum into an integral in polar coordinates and carried out the angular integral (over a quarter-circle). But now, changing variables to x (which is proportional to n^2) gives the integral

$$N_m \propto \int_0^\infty \frac{1}{e^x - 1} dx.$$

Near $x = 0$, we can expand $e^x \approx 1 + x + \dots$ and cancel the 1 to see that the integrand is proportional to $1/x$. Therefore, the integral diverges at its lower limit; this implies that the number of long-wavelength magnons is infinite. The only obvious way out of this contradiction is to suppose that the material doesn't magnetize in the first place, and this turns out to be true.

Problem 7.65. To evaluate the integral I used the *Mathematica* instruction

```
NIntegrate[Sqrt[x]/(Exp[x]-1), {x, 0, Infinity}]
```

and it returned 2.31516, confirming the value quoted in the text.

Problem 7.66. (Bose-Einstein condensation of rubidium-87.)

- (a) For a rubidium-87 atom in a cube-shaped box of with 10^{-5} m, the ground-state energy is

$$\begin{aligned} \epsilon_0 &= \frac{h^2}{8mL^2} (1^2 + 1^2 + 1^2) = \frac{3}{8} \frac{(6.63 \times 10^{-34} \text{ J}\cdot\text{s})^2}{(87)(1.66 \times 10^{-27} \text{ kg})(10^{-5} \text{ m})^2} \\ &= 1.14 \times 10^{-32} \text{ J} = 7.1 \times 10^{-14} \text{ eV}. \end{aligned}$$

This is a tiny energy indeed.

- (b) According to equation 7.126, the condensation temperature is

$$kT_c = (0.527) \left(\frac{h^2}{2\pi mL^2} \right) N^{2/3} = (0.224) N^{2/3} \epsilon_0,$$

where the coefficient 0.224 comes from comparing this expression to the previous one. If there are 10,000 atoms in our box, then the kT_c is greater than ϵ_0 by a factor of $(0.224)(10,000)^{2/3} = 104 \approx 100$, that is, $kT_c = 7.4 \times 10^{-12}$ eV or $T_c = 8.6 \times 10^{-8}$ K. This is in rough agreement with the value 10^{-7} K quoted on page 319.