Physics 7602 Final Exam – Spring 2019

- *Closed-book exam:* no notes, no phone
- Please begin each of the 4 problems on a new page.

Problem 1. (30 points)

The Landau free energy per site $f(m,t,h)$ for a uniaxial magnet in 3 dimensions can be expressed as a function of reduced variables m, t, and h for the magnetization, temperature, and magnetic field. Near the critical point ($m=0$, $t=0$, $h=0$), $f(m,t,h)$ satisfies the scaling equation

$$f(m,t,h) - f(0,0,0) = \lambda^3\left[f\left(\lambda^{\Delta_M}m, \lambda^{\Delta_T}t, \lambda^{\Delta_H}h\right) - f(0,0,0)\right],$$

where Δ_M, Δ_T, and Δ_H are anomalous scaling dimensions.

The equation of state can be expressed as $m = \partial f/\partial h$.

5 pts A. Deduce Δ_M as a function of Δ_T and Δ_H.

$$m = \frac{\partial f}{\partial h}, \quad \lambda^{\Delta_M} m = \frac{\partial(\lambda^3 f)}{\partial(\lambda^{\Delta_H} h)} \implies \lambda^{\Delta_M} = \lambda^{3-\Delta_H} \qquad \Delta_M = 3 - \Delta_H$$

The critical exponent δ for the critical isotherm can be defined by $m \sim h^{1/\delta}$ as $h \to 0^+$ with $t = 0$.

5 pts B. Deduce δ as a function of Δ_T and Δ_H (or Δ_M).

$$m \sim h^{1/\delta} \quad \lambda^{\Delta_M} m \sim (\lambda^{\Delta_H} h)^{1/\delta} \implies \lambda^{\Delta_M} = \lambda^{\Delta_H/\delta} \qquad \delta = \Delta_H/\Delta_M = \frac{\Delta_H}{3-\Delta_H}$$

Near the critical point, the correlator $G(r;t,h)$ for spins separated by a distance r that is much larger than the lattice spacing satisfies the scaling relation

$$G(r;t,h) = \frac{1}{\lambda^{2\Delta_M}} G\left(r/\lambda; \lambda^{\Delta_T}t, \lambda^{\Delta_H}h\right).$$

The critical exponent ν is defined by the behavior of the correlation length $\xi(t,h)$ near the critical point: $\xi \sim 1/t^\nu$ as $t \to 0^+$ with $h = 0$.

5 pts C. Deduce ν as a function of Δ_T and Δ_H (or Δ_M).

$$\exp\left(-r/\xi(t,h)\right) = \exp\left(-(r/\lambda)/\xi(\lambda^{\Delta_T}t, \lambda^{\Delta_H}h)\right)$$

$$\implies \xi(t,h) = \lambda\, \xi(\lambda^{\Delta_T}t, \lambda^{\Delta_H}h)$$

$$\frac{1}{t^\nu} = \lambda \frac{1}{(\lambda^{\Delta_T}t)^\nu} \implies 1 = \lambda^{1-\Delta_T\nu} \qquad \nu = \frac{1}{\Delta_T}$$

The Hamiltonian for the Ising model with spin variables $\sigma_i = \pm 1$ on lattice sites i with spacing a at 0 magnetic field can be expressed as

$$\beta H = -K(t;a) \sum_{\langle ij \rangle} \sigma_i \sigma_j,$$

where t is the reduced temperature and the sum is over nearest neighbors.
An RG transformation produces another Ising model with spin variables $\sigma_i = \pm 1$ on lattice sites i with a larger spacing λa. The effective Hamiltonian can be expressed as

$$\beta H_{\text{eff}} = \beta F_0(t;\lambda a) - K(t;\lambda a) \sum_{\langle ij \rangle} \sigma_i \sigma_j.$$

The RG equation expresses $K(t;\lambda a)$ as a function of $K(t;a)$:

$$K(t;\lambda a) = R(K(t;a)).$$

This RG equation has a fixed point K_*.

3 pts D. Expand the right side of the RG equation around the fixed point to first order in $K - K_*$.

$$R(K(t;a)) = R(K_*) + R'(K_*) \cdot [K(t;a) - K_*]$$
$$= K_* + R'(K_*) \cdot [K(t;a) - K_*]$$

Near the critical temperature $t = 0$, the Ising coupling constant has the scaling behavior

$$K(t;\lambda a) = K(\lambda^{\Delta_T} t; a).$$

5 pts E. Use the scaling relation to eliminate $K(t;\lambda a)$ from the RG equation. Expand each side of the RG equation to first order in t.

$$K(\lambda^{\Delta_T} t; a) = R(K(t;a))$$
$$K(\lambda^{\Delta_T} t; a) = K(0,a) + \frac{\partial K}{\partial t}(0;a) \cdot \lambda^{\Delta_T} t = K_* + \frac{\partial K}{\partial t}(0;a) \cdot \lambda^{\Delta_T} t$$
$$R(K(t;a)) = K_* + R'(K_*) \cdot \frac{\partial K}{\partial t}(0;a) \cdot t$$

4 pts F. Use the RG equation to deduce a formula for the anomalous dimension Δ_T in terms of the function $R(K)$.

$$\frac{\partial K}{\partial t}(0;a) \cdot \lambda^{\Delta_T} t = R'(K_*) \frac{\partial K}{\partial t}(0;a)\, t \implies \lambda^{\Delta_T} = R'(K_*)$$

$$\Delta_T = \frac{\log R'(K_*)}{\log \lambda}$$

For the 2-dimensional Ising model, there is an RG transformation that changes the lattice spacing from a to $\sqrt{2}\,a$. The RG function near the fixed point K_* is

$$R(K) = K_* + \tfrac{3}{2} \tanh(4K_*)(K - K_*) + \ldots.$$

3 pts G. Express the anomalous dimension Δ_T as a function of K_*.

$$\lambda = \sqrt{2} \qquad R'(K_*) = \tfrac{3}{2} \tanh(4K_*) \qquad \Delta_T = \frac{\log\left(\frac{3}{2} \tanh(4K_*)\right)}{\log \sqrt{2}}$$

Problem 2. (20 points)

If a massless boson is in thermal equilibrium at temperature T and in chemical equilibrium with chemical potential 0, the number density, energy density, and entropy density of a single spin state are

$$n_b = \frac{\zeta_3}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3, \qquad u_b = \frac{\pi^2}{30}\left(\frac{kT}{\hbar c}\right)^3 kT, \qquad s_b = \frac{2\pi^2}{45}\left(\frac{kT}{\hbar c}\right)^3.$$

If a massless fermion is in thermal equilibrium at temperature T and in chemical equilibrium with chemical potential 0, the number density, energy density, and entropy density of a single spin state are

$$n_f = \tfrac{3}{4}n_b, \qquad u_f = \tfrac{7}{8}u_b \qquad s_f = \tfrac{7}{8}s_b.$$

Suppose the only particles in the early universe at time t_1 are electrons, positrons, and photons. They form an electrically neutral plasma in thermal equilibrium at a temperature T_1 that is much larger than the electron mass m_e.

3 pts A. Identify the numbers of spin states of the photon, the electron, and the positron.

photon: 2 polarizations $\Rightarrow$ 2 spin states

electron: spin $\frac{1}{2}$ $\Rightarrow$ 2 spin states

positron: spin $\frac{1}{2}$ $\Rightarrow$ 2 spin states

3 pts B. What is the total energy density of the universe at time t_1?

$$u(t_1) = 2u_b(T_1) + (2+2)u_f(T_1)$$

$$= \left[2 + 4\cdot\tfrac{7}{8}\right]u_b(T_1) = \frac{11}{2}\frac{\pi^2}{30}\left(\frac{kT_1}{\hbar c}\right)^3 kT_1$$

3 pts C. What is the ratio of the number densities of electrons and photons at time t_1?

$$n_{e^-}(t_1) = 2n_f(T_1) \qquad \frac{n_{e^-}(t_1)}{n_\gamma(t_1)} = \frac{2n_f(T_1)}{2n_b(T_1)} = \frac{\frac{3}{4}n_b(T_1)}{n_b(T_1)} = \frac{3}{4}$$

$$n_\gamma(t_1) = 2n_b(T_1)$$

The universe expands until a time t_2 when its temperature T_2 is much smaller than m_e. The electrons and positrons have all annihilated.

3 pts D. What is the total energy density of the universe at time t_2?

$$u(t_2) = 2n_b(T_2) = 2\cdot\frac{\pi^2}{30}\left(\frac{kT_2}{\hbar c}\right)^3 kT_2$$

The expansion of the universe changes a length L at time t_a to a length at time t_b that is larger by a factor of $a(t_b)/a(t_a)$, where $a(t)$ is the scale factor at time t.

The expansion of the universe changes its temperature from $T_1 \gg m_e$ at time t_1 to $T_2 \ll m_e$ at time t_2.

4 pts E. Express the ratio $a(t_2)/a(t_1)$ of scale factors in terms of the temperatures T_2 and T_1.

constant entropy $\Longrightarrow$ $s(t)\,a(t)^3$ is constant

$$s(t_1) = 2 s_b(T_1) + (2+2) s_f(T_1) = \left[2 + 4\cdot\tfrac{7}{8}\right] s_b(T_1) \qquad s(t_2) = 2 \cdot s_b(T_2)$$

$$s(t_1)a(t_1)^3 = s(t_2)\,a(t_2)^3 \qquad \left(\frac{a(t_2)}{a(t_1)}\right)^3 = \frac{s(t_1)}{s(t_2)} = \frac{\frac{11}{2} s_b(T_1)}{2 s_b(T_2)} = \frac{11}{4}\left(\frac{T_1}{T_2}\right)^3$$

The universe expands further until the present time t_4, when its temperature is $T_4 = 2.7K$ and its scale factor is $a(t_4) = 1$.

$(t_3 > t_2)$

4 pts F. What was the scale factor $a(t_3)$ at the time t_3 when the temperature was T_3?

$$\boxed{\frac{a(t_2)}{a(t_1)} = \left(\frac{11}{4}\right)^{1/3} \frac{T_2}{T_1}}$$

$t_3 > t_2$

$$s(t_3)\,a(t_3)^3 = s(t_4)\,a(t_4)^3$$

$$s(t_3) = 2\, s_b(T_3) \qquad s(t_4) = 2\, s_b(T_4)$$

$$a(t_3) = \left(\frac{s(t_4)}{s(t_3)}\right)^{1/3} a(t_4) = \left(\frac{2 s_b(T_4)}{2 s_b(T_3)}\right)^{1/3} \cdot 1$$

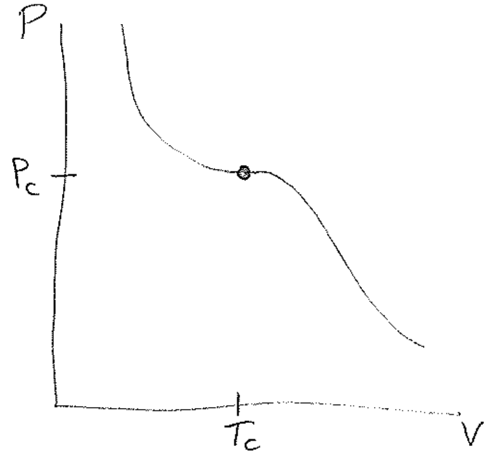
$$= \left(\frac{T_4^3}{T_3^3}\right)^{1/3} = \frac{T_4}{T_3} = \frac{2.7K}{T_3}$$

Problem 3. (30 points)

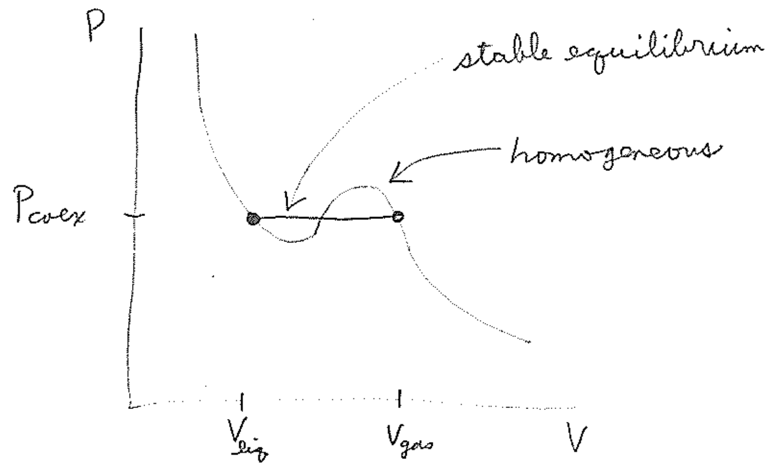
The van der Waals equation of state that describes a liquid/gas phase transition is

$$\left(P + a\frac{N^2}{V^2}\right)(V - bN) = nkT.$$

3 pts A. Sketch a homogeneous isotherm in the $V - P$ plane for the critical temperature T_c where the phase transition ends. Label the critical volume V_c and the critical pressure P_c.



4 pts B. Sketch the isotherm in the $V - P$ plane for a temperature T below T_c. Sketch the stable equilibrium isotherm at the same temperature T.



The coexistence pressure $P_{coex}(T)$ on the stable equilibrium isotherm can be determined by considering line integrals $\int P(V)\,dV$ along paths in the $V - P$ plane.

5 pts C. Write down the equation that determines $P_{coex}(T)$, specifying clearly the endpoints and paths of the line integrals.

integral from (V_{liq}, P_{coex}) to (V_{gas}, P_{coex}) along horizontal path is equal to integral along homogenous isotherm

$$P_{coex}(T)\cdot\left[V_{gas}(T) - V_{gas}(T)\right] = \int_{V_{liq}(T)}^{V_{gas}(T)} P(V)\,dV$$

3 pts D. State the implications of the equation for $P_{coex}(T)$ in terms of areas in the $V - P$ plane in part B.

areas of the two regions between the homogenous isotherm and the horizontal line must be equal

The van der Waals equation of state can be expressed in terms of dimensionless reduced variables $p = (P - P_c)/P_c$, $v = (V - V_c)/V_c$, and $t = (T - T_c)/T_c$. It can then expanded in powers of v:

$$p = -(\tfrac{3}{2}v^3 - \tfrac{21}{4}v^4 + \ldots) + t(4 - 6v + 9v^2 - \tfrac{27}{2}v^3 + \ldots).$$

Near the critical point, the equation of state can be reduced to

$$p - 4t = -\tfrac{3}{2}v(v^2 + 4t).$$

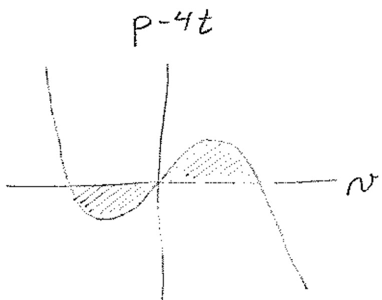
4pts E. Determine how p, t, and $p - 4t$ must scale with v in order that all other terms be suppressed near the critical point.

$v^3 \sim v^2 t \implies t \sim v^2$

$p - 4t \sim v^3 \implies p - 4t \sim v^3$

$t \sim v^2 \implies p \sim v^2$

4pts F. Identify the coexistence pressure $p_{\mathrm{coex}}(t)$ at which the liquid and gas can be in equilibrium at a temperature $t < 0$.



equal areas $\implies p_{coex}(t) - 4t = 0$

$p_{coex}(t) = 4t$

Suppose the system is at a temperature $t < 0$ with pressure $p_{\mathrm{coex}}(t)$.

4 pts G. Determine the volumes $v_{\mathrm{liq}}(t)$ and $v_{\mathrm{gas}}(t)$ for the equilibrium states that consist of 100% liquid and 100% gas, respectively.

$p_{coex}(t) = 4t \implies -\frac{3}{2}v(v^2 + 4t) = 0$

$v_{liq}(t), v_{gas}(t)$ satisfy $v^2 + 4t = 0 \implies v = \pm\sqrt{-4t}$

$v_{liq}(t) < v_{gas}(t) \implies v_{liq}(t) = -2\sqrt{-t}, \; v_{gas}(t) = +2\sqrt{-t}$

The critical exponent β can be defined by $v_{\mathrm{gas}} \sim |t|^\beta$ as $t \to 0^-$.

3pts H. Determine β.

$v_{gas}(t) = 2\sqrt{|t|} = 2|t|^{1/2} \implies \beta = \frac{1}{2}$

Problem 4. (20 points)

In the early universe, a particle that is massless or ultrarelativistic has chemical potential 0. The chemical potential μ_i for a nonrelativistic particle i with mass m_i and spin s_i that has number density n_i and is in thermal equilibrium at temperature T is

$$\mu_i = m_i c^2 + kT\left[\log(n_i\lambda_i^3) - \log(2s_i+1)\right], \qquad \lambda_i = \sqrt{2\pi\hbar^2/m_i kT}.$$

When kT is much smaller than the proton mass m_p and much larger than the electron mass m_e, the proton and neutron are in chemical equilibrium with ultrarelativistic electrons and neutrinos through electroweak reactions, such as $n \longleftrightarrow p e^- \nu_e$ $\bar{\nu}_e$, $n e^+ \longleftrightarrow p \nu_e$ $\bar{\nu}_e$, ...

3 pts A. Deduce the relation between the chemical potentials of the proton and neutron.

$$ne^+ \longleftrightarrow p\bar{\nu}_e \implies \mu_n + 0 = \mu_p + 0 \implies \mu_n = \mu_p$$

5 pts B. Determine the ratio of the number densities of the proton and neutron as a function of T.

$$n_n = m_n c^2 + kT[\log(n_n\lambda_n^3) - \log 2]$$
$$n_p = m_p c^2 + kT[\log(n_p\lambda_p^3) - \log 2]$$
$$0 = (m_n - m_p)c^2 + kT\log\frac{n_n\lambda_n^3}{n_p\lambda_p^3} \qquad m_p \simeq m_n$$
$$\frac{n_n}{n_p} = \frac{\lambda_p^3}{\lambda_n^3}e^{-(m_n-m_p)c^2/kT} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-(m_n-m_p)c^2/kT} \simeq e^{-(m_n-m_p)c^2/kT}$$

When kT is comparable to m_e, the electron and positron are in chemical equilibrium with photons through electromagnetic reactions, such as $e^+e^- \longleftrightarrow \gamma\gamma$.

3 pts C. Deduce the relation between the chemical potentials of the electron and positron.

$$e^+e^- \longleftrightarrow \gamma\gamma \implies \mu_{e^+} + \mu_{e^-} = 2\cdot 0 \implies \mu_{e^+} = -\mu_{e^+}$$

During nucleosynthesis, the proton, neutron, and deuteron (bound state of pn with spin 1) are in chemical equilibrium through the reaction $pn \longleftrightarrow d\gamma$.

3 pts D. Deduce the relation between the chemical potentials of the proton, neutron, and deuteron.

$$pn \longleftrightarrow d\gamma \implies \mu_p + \mu_n = \mu_d + 0 \implies \mu_d = \mu_p + \mu_n$$

6 pts E. Express the number density of the deuteron in terms of the number densities of the proton and neutron and the temperature T.

$$\mu_d = m_d c^2 + kT[\log(n_d\lambda_d^3) - \log 3]$$
$$\mu_p = m_p c^2 + kT[\log(n_p\lambda_p^3) - \log 2]$$
$$\mu_n = m_n c^2 + kT[\log(n_n\lambda_n^3) - \log 2]$$
$$0 = (m_d - n_p - m_n)c^2 + kT\left[\log\frac{n_d\lambda_d^3}{n_p\lambda_p^3 n_n\lambda_n^3} - \log 3 + 2\log 2\right]$$
$$\frac{n_d\lambda_d^3}{n_p\lambda_p^3 n_n\lambda_n^3}\,\frac{4}{3} = e^{-(m_d-m_n-n_p)c^2/kT} \qquad n_d = \frac{3}{4}n_p n_n\lambda_d^3\left(\frac{m_d^2}{m_n m_p}\right)^{3/2} e^{-(m_d-m_p-m_n)c^2/kT}$$
$$m_d \simeq 2m_p \simeq 2m_n \qquad \simeq 6\, n_p n_n\left(\frac{\pi\hbar^2}{m_p kT}\right)^{3/2} e^{-(m_d-m_n-m_p)c^2/kT}$$