**Microcanonical Ensemble**

microcanonical ensemble with energy $E$

all microstates with energy $E$

have equal probabilities

other microstates have probability 0

thermodynamic variables can be deduced

from a single function of $E$

and other variables specifying macrostate $(V, N)$

$\Omega(E) = \text{number of microstates with energy } E$

probability distribution

$$P_i = \frac{1}{\Omega(E)} \quad \text{if } E_i = E$$

$$= 0 \quad \text{if } E_i \neq E$$

energy averaged over ensemble

$$\langle E \rangle = \sum_i P_i E_i = \sum_{E_i = E} \frac{1}{\Omega(E)} E_i$$

$$= \frac{\Omega(E) E}{\Omega(E)} = E$$

$$= \sum_{E_i = E} \frac{1}{\Omega(E)} \Omega(E)$$
Information entropy:

\[ S_{\text{info}} = - \sum_i P_i \log P_i \]

restrict sum to states with nonzero \( P_i \):

\[ = - \sum_{i : E_i = E} P_i \log P_i \]

\[ = - \sum_{i : E_i = E} \frac{1}{\Omega(E)} \log \frac{1}{\Omega(E)} \]

\[ = - \frac{1}{\Omega(E)} \log \frac{1}{\Omega(E)} \sum_{i : E_i = E} \frac{1}{\Omega(E)} \]

\[ = - \frac{1}{\Omega(E)} \log \frac{1}{\Omega(E)} \cdot \Omega(E) \]

\[ = - \log \frac{1}{\Omega(E)} \]

\[ = \log \Omega(E) \]

Classical system:

\[ \Omega(E) = \int d\omega \ S(H(\rho,\varphi) - E) \Delta E \]

= volume of phase space with energy \( H(\rho,\varphi) \) between \( E - \frac{1}{2} \Delta E \) and \( E + \frac{1}{2} \Delta E \)
for macroscopic system,

\[ \Omega(E) \text{ increases extremely rapidly with } E \]

example of such a function: \( E^{N_A} \), where \( N_A = 6 \times 10^{23} \)

graph of \( \Omega(E) \):

\[
\begin{array}{c|c|c|c|c}
\Omega(E) \\
\hline
E
\end{array}
\]

graph of \( \log \Omega(E) \):

\[
\begin{array}{c|c}
\log \Omega(E) \\
E
\end{array}
\]

\( \log \Omega(E) \text{ increases roughly like } \log E \)
Two macroscopic systems A and B...

in thermal contact

but otherwise isolated

energies: $E_A$ and $E_B$

\[ E_A \leftrightarrow E_B \]

A

B

thermal contact: can exchange energy

otherwise isolated $\Rightarrow$ total energy $E_A + E_B$ is conserved

number of microstates

for A: $\Omega_A(E_A)$

for B: $\Omega_B(E_B)$

for A and B: $\Omega_A(E_A) \times \Omega_B(E_B)$

Microcanonical ensemble with total energy $E$

all microstates with $E_A + E_B = E$

are equally likely

$$\Omega_{A+B}(E, E_A) = \Omega_A(E_A) \cdot \Omega_B(E - E_A)$$
energy $E_A$ is overwhelmingly likely
to be extremely near
the value that maximizes $\Omega_A(E_A) \Omega_B(E-E_A)$

information entropy

$$S_{A+B}(E, E_A) = S_A(E_A) + S_B(E-E_A)$$

energy $E_A$ is overwhelmingly likely
to be extremely near
the value that maximizes total entropy
maximize total entropy:

\[
\frac{d}{dE_A} \left( S_A(E_A) + S_B(E-E_A) \right) = 0
\]

\[
\frac{dS_A}{dE_A}(E_A) + \frac{dS_B}{dE_B}(E-E_A) \cdot \frac{d}{dE_A}(E-E_A) = 0
\]

\[
\frac{dS_A}{dE_A}(E_A) - \frac{dS_B}{dE_B}(E-E_A) = 0
\]

equilibrium condition:

\[
\frac{\partial S_A}{\partial E_A} = \frac{\partial S_B}{\partial E_B}
\]

from thermodynamics:

\[
T_A = T_B
\]

\[\Rightarrow \quad \frac{\partial S}{\partial E} \text{ must be determined by temperature } T\]

fundamental thermodynamic relation

\[
dE = TdS - PdV + \ldots
\]

\[\Rightarrow \, T = \left( \frac{\partial E}{\partial S} \right)_V, \ldots\]

If information entropy and thermodynamic entropy can be identified, then

\[
\left( \frac{\partial S}{\partial E} \right)_V = \frac{1}{T}
\]
Two macroscopic systems A and B that can exchange volume with \( V = V_A + V_B \) fixed:

\[
V_A \leftrightarrow V_B
\]

\[ A \quad B \]

Equilibrium condition:

\[
\frac{\partial S_A}{\partial V_A} = \frac{\partial S_B}{\partial V_B}
\]

Mechanics: \( P_A = P_B \) at equilibrium

\[ \Rightarrow \frac{\partial S}{\partial V} \text{ must be determined by pressure } P \]

Thermodynamics:

\[ dF = T dS - P dV + \ldots \]

\[ \Rightarrow P = T \left( \frac{\partial S}{\partial V} \right)_F, \ldots \]

If information entropy and thermodynamic entropy can be identified then

\[ \left( \frac{\partial S}{\partial V} \right)_F, \ldots = \frac{P}{T} \]