Trapped Bosonic Atoms

bosonic atoms	isotope	nuclear spin i
	^{87}Rb	$\frac{3}{2}$
	^{23}Na	$\frac{3}{2}$
	^{133}Cs	$\frac{7}{2}$
	H	$\frac{1}{2}$

nuclear spin: $\vec{I}$ with quantum number i

electron spin: $\vec{S}$ with quantum number $s = \frac{1}{2}$

hyperfine spin: $\vec{F} = \vec{I} + \vec{S}$

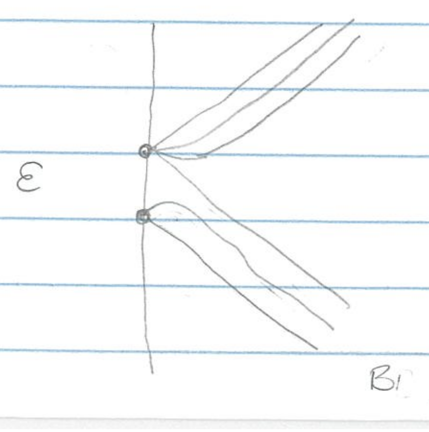
quantum number: $f = |i-s|, \ldots, i+s$

hyperfine interaction: $H_{hf} = \frac{\varepsilon_{hf}}{\hbar^2} \vec{I} \cdot \vec{S}$

hyperfine states: $|f, m_f\rangle$, $m_f = -f, \ldots, +f$

energy levels: $\varepsilon_f = \frac{\varepsilon_{hf}}{\hbar^2} \frac{f(f+1) - i(i+1) - s(s+1)}{2}$

magnetic field: $H_{mag} = -\mu_B \vec{B} \cdot \vec{S}$



atoms trapped at center of vacuum chamber

harmonic trapping potential

$$V(x,y,z) = \frac{1}{2} m\omega^2 (x^2+y^2+z^2)$$

$$V(\vec{r}) = \frac{1}{2} m\omega^2 \vec{r}^{\,2}$$

magnetic trap (using magnetic fields):

lowest hyperfine state, $|f,-f\rangle$
single spin state

optical trap (using laser beams)

lowest hyperfine multiplet $f=|i-s|$
$2f+1$ spin state

atoms cooled to ultralow temperatures.

laser cooling $\longrightarrow$ mK
evaporative cooling $\longrightarrow$ nK

single-particle states

energy levels: $\varepsilon_i = \varepsilon_{n_x, n_y, n_z} = (n_x + n_y + n_z + \frac{3}{2})\hbar\omega$

$$n_x, n_y, n_z = 0, 1, 2, \ldots$$

lowest energy state: $\varepsilon_0 = \frac{3}{2}\hbar\omega$

$$\psi_0(x,y,z) = \frac{1}{(\sqrt{\pi}\, a)^{3/2}} \exp\left(-\frac{x^2+y^2+z^2}{2a^2}\right)$$

$$a = \sqrt{\hbar/m\omega}$$

classical limit:

energies: $\varepsilon_i \longrightarrow \varepsilon(\vec{r}, \vec{p}) = \frac{1}{2m}\vec{p}^{\,2} + \frac{1}{2}m\omega^2\vec{r}^{\,2}$

sum over single particle states:

$$\sum_i \longrightarrow \frac{1}{(2\pi\hbar)^3}\int d^3r \int d^3p$$

Grand Canonical Ensemble with temperature $T = 1/\beta$

chemical potential μ

occupation numbers:

ground state: N_0

higher energy state: $\dfrac{1}{e^{\beta[\epsilon(\vec{r},\vec{p}) - \mu]} - 1}$

atom number

above T_c: $N = \frac{1}{(2\pi\hbar)^3} \int d^3r \int d^3p \frac{1}{\exp(\beta[\varepsilon(\vec{r},\vec{p}) - \mu]) - 1}$

below T_c: $N = N_0 + \frac{1}{(2\pi\hbar)^3} \int d^3r \int d^3p \frac{1}{\exp(\beta\varepsilon(\vec{r},\vec{p})) - 1}$

to evaluate integral

1. change variables from $\vec{r} = (x, y, z)$, $\vec{p} = (p_x, p_y, p_z)$
to 6-dimensional vector $\vec{P} = (P_1, P_2, P_3, P_4, P_5, P_6)$

$$P_1 = p_x \qquad P_2 = p_y \qquad P_3 = p_z$$

$$P_4 = m\omega x \qquad P_5 = m\omega y \qquad P_6 = m\omega z$$

$$\varepsilon(\vec{p},\vec{r}) = \frac{1}{2m}\left(P_1^2 + P_2^2 + P_3^2 + P_4^2 + P_5^2 + P_6^2\right) = \frac{1}{2m}\vec{P}^2$$

$$\int d^3r \int d^3p = \frac{1}{(m\omega)^3} \int d^6P$$

2. spherical coordinates

$$\int d^6P = \int P^5 \, dP \, d\Omega_6 = \Omega_6 \int_0^\infty P^5 \, dP \qquad \Omega_6 = \pi^3$$

3. energy: $\varepsilon = \frac{1}{2m}P^2 \qquad d\varepsilon = \frac{1}{m} P\,dP$

$$\int_0^\infty P^5 \, dP = \frac{1}{2}\left(\sqrt{2m}\right)^6 \int_0^\infty \varepsilon^2 \, d\varepsilon$$

4. dimensionless variable: $x = \beta \varepsilon$

$$\int \varepsilon^2 d\varepsilon = \frac{1}{\beta^3}\int_0^\infty x^2 dx = T^3 \int_0^\infty x^2 dx$$

$$\frac{1}{(2\pi\hbar)^3}\int d^3r \int d^3p = \frac{1}{(2\pi\hbar)^3}\frac{1}{2(m\omega)^3}\pi^3 (2m)^3 \cdot T^3 \int_0^\infty x^2 dx$$

$$= \frac{1}{2}\left(\frac{T}{\hbar\omega}\right)^3 \int_0^\infty x^2 dx$$

Bose-Einstein function: $g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty dx\, x^{\nu-1} \frac{1}{\frac{1}{z}e^x - 1}$

$$g_\nu(1) = \zeta_\nu$$

<u>atom number</u>

above T_c: $N = \left(\frac{T}{\hbar\omega}\right)^3 g_3(z) \qquad z = e^{\beta\mu}$

below T_c: $N = N_0 + \zeta_3 \left(\frac{T}{\hbar\omega}\right)^3$

critical temperature: $N = \zeta_3\left(\frac{T_c}{\hbar\omega}\right)^3 \Longrightarrow T_c = \left(\frac{N}{\zeta_3}\right)^{1/3}\hbar\omega$

express number of thermal atoms in terms of T/T_c

$$\zeta_3\left(\frac{T}{\hbar\omega}\right)^3 = N\left(\frac{T}{T_c}\right)^3$$

solve for number of condensed atoms: $N_0(T) = N\left[1 - \left(\frac{T}{T_c}\right)^3\right]$

Spatial distribution:

local number density: $n(\vec{r})$

$$N = \int d^3r \, n(\vec{r})$$

$T \geq T_c$: $N = \frac{1}{(2\pi\hbar)^3} \int d^3r \int d^3p \, \frac{1}{e^{\beta(\varepsilon(p,r)-\mu)} - 1}$

$$\Longrightarrow n(\vec{r}) = \frac{1}{(2\pi\hbar)^3} \int d^3p \, \frac{1}{e^{\beta(\varepsilon(p,r)-\mu)} - 1}$$

$T \gg T_c$: $n(\vec{r}) \simeq \frac{1}{(2\pi\hbar)^3} e^{\beta\mu} e^{-\beta(m\omega^2 r^2/2)} \int d^3p \, e^{-\beta(p^2/2m)}$

$$= \frac{e^{\beta\mu}}{\lambda_T^3} e^{-\beta m\omega^2 r^2/2} \qquad \lambda_T = \sqrt{\frac{2\pi\hbar^2}{mT}}$$

$\Longrightarrow$ size of thermal cloud: $\langle r \rangle \sim \frac{1}{\sqrt{\beta m\omega^2}} = \sqrt{\frac{T}{m\omega^2}}$

$T = T_c$: $n(\vec{r}) \simeq \frac{1}{(2\pi\hbar)^3} \int d^3p \, \frac{1}{e^{\beta_c \varepsilon(p,r)} - 1}$

$$= \frac{1}{\lambda_{T_c}^3} g_{3/2}\left(e^{-\beta_c m\omega^2 r^2/2}\right)$$

$\Longrightarrow$ size of thermal cloud: $\langle r \rangle \sim \frac{1}{\sqrt{\beta_c m\omega^2}} = \sqrt{\frac{T_c}{m\omega^2}}$

$$\sim N^{1/3} \sqrt{\frac{\hbar}{m\omega}}$$

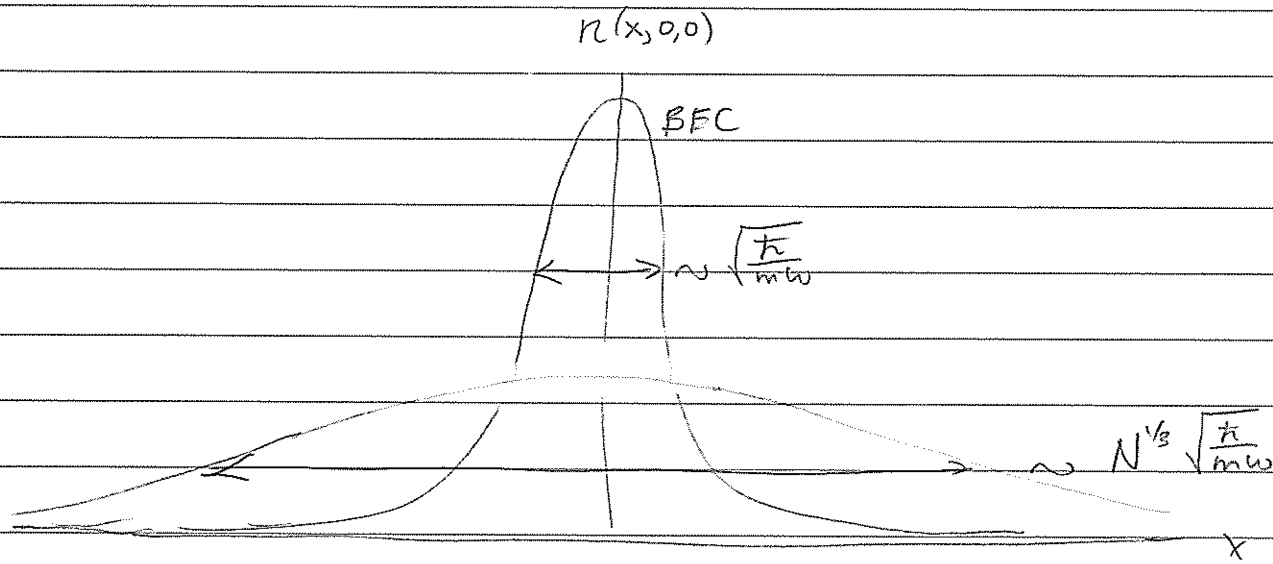
$\underline{T=0}$: all atoms in ground state with wavefunction

$$\psi_0(\vec{r}) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-(m\omega/\hbar)r^2/2}$$

number density: $n(\vec{r}) = N\,|\psi_0(\vec{r})|^2$

$$= N\left(\frac{m\omega}{\pi\hbar}\right)^{3/2} e^{-(m\omega/\hbar)r^2}$$

size of condensate: $\langle r\rangle \sim \sqrt{\frac{\hbar}{m\omega}}$



radius of thermal cloud is larger than that of condensate by a factor of order $N^{1/3}$

Momentum Distribution

$$N = \int d^3p \, n(\vec{p})$$

anisotropic trap: $V(x,y,z) = \frac{1}{2} m \omega_\perp^2 (x^2+y^2) + \frac{1}{2} m \omega_\parallel^2 z^2$

above T_c:

$$n(\vec{p}) = \frac{1}{(2\pi\hbar)^3} \int d^3r \, \frac{1}{\exp\left(-\beta\left[\frac{1}{2m}\vec{p}^2 + V(\vec{r}) - \mu\right]\right) - 1}$$

isotropic

at $T=0$: $\psi_0(\vec{r}) = \frac{1}{\sqrt{\pi}\, a_\perp} \exp\left(-\frac{x^2+y^2}{2a_\perp^2}\right) \frac{1}{(\sqrt{\pi}\, a_\parallel)^{1/2}} \exp\left(-\frac{z^2}{2a_\parallel^2}\right)$

$$\psi_0(\vec{p}) = \frac{a_\perp}{\sqrt{\pi}\,\hbar} \exp\left(-\frac{a_\perp^2 p^2}{2\hbar^2}\right) \left(\frac{a_\parallel}{\sqrt{\pi}\,\hbar}\right)^{1/2} \exp\left(-\frac{a_\parallel^2 p^2}{2\hbar^2}\right)$$

$$n(\vec{p}) = N \left|\psi_0(\vec{p})\right|^2$$

$$= \frac{N a_\perp^2 a_\parallel}{\pi^{3/2} \hbar^3} \exp\left(-\frac{a_\perp^2 p^2}{\hbar^2}\right) \exp\left(-\frac{a_\parallel^2 p^2}{\hbar^2}\right)$$