Relations between Partial Derivatives

Suppose the variables $X$, $Y$, and $Z$ are related so that only two of them are independent. The differential of any one of them can be expressed in terms of differentials of the other two:

$$dX = \left( \frac{\partial X}{\partial Y} \right)_Z dY + \left( \frac{\partial X}{\partial Z} \right)_Y dZ,$$

$$dY = \left( \frac{\partial Y}{\partial X} \right)_Z dX + \left( \frac{\partial Y}{\partial Z} \right)_X dZ.$$

A. Insert the expression for $dY$ into the equation for $dX$ and collect the coefficients of $dX$ and $dZ$ on the right side.

$$dX = \left( \frac{\partial X}{\partial Y} \right)_Z \left[ \left( \frac{\partial Y}{\partial X} \right)_Z dX + \left( \frac{\partial Y}{\partial Z} \right)_X dZ \right] + \left( \frac{\partial X}{\partial Z} \right)_Y dZ$$

$$= \left( \frac{\partial X}{\partial Y} \right)_Z \left( \frac{\partial Y}{\partial X} \right)_Z dX + \left[ \left( \frac{\partial X}{\partial Y} \right)_Z \left( \frac{\partial Y}{\partial Z} \right)_X + \left( \frac{\partial X}{\partial Z} \right)_Y \right] dZ$$

The independence of the variables $X$ and $Z$ implies that the coefficients of $dX$ and $dZ$ on both sides of the equation must match.

B. Write down the resulting two equations involving partial derivatives.

$$1 = \left( \frac{\partial X}{\partial Y} \right)_Z \left( \frac{\partial Y}{\partial X} \right)_Z$$

$$0 = \left( \frac{\partial X}{\partial Y} \right)_Z \left( \frac{\partial Y}{\partial Z} \right)_X + \left( \frac{\partial X}{\partial Z} \right)_Y$$

C. Deduce the identity

$$\left( \frac{\partial Y}{\partial X} \right)_Z = \frac{1}{\left( \frac{\partial X}{\partial Y} \right)_Z}.$$

1st equation:

$$\left( \frac{\partial X}{\partial Y} \right)_Z = \frac{1}{\left( \frac{\partial Y}{\partial X} \right)_Z}.$$

D. Express $(\partial X/\partial Y)_Z$ in terms of a ratio of partial derivatives with respect to $Z$.

2nd equation:

$$\left( \frac{\partial X}{\partial Y} \right)_Z = -\frac{\left( \frac{\partial X}{\partial Z} \right)_Y}{\left( \frac{\partial Y}{\partial Z} \right)_X}$$

$$= -\frac{1}{\left( \frac{\partial Z}{\partial Y} \right)_X} \frac{\partial X}{\partial Z}_Y = -\frac{\partial X/\partial Z}_Y \frac{\partial Y/\partial Z}_X$$

Compare your results with your neighbors.

Show your results to the instructor.
The energy $E$, volume $V$, and entropy $S$ of a system are related by the equilibrium condition. In the microcanonical ensemble, the temperature $T$ and the pressure $P$ are naturally expressed in terms of derivatives of $S$:

$$\frac{1}{T} = \left( \frac{\partial S}{\partial E} \right)_V, \quad P = \left( \frac{\partial S}{\partial V} \right)_E.$$

E. Use relations between partial derivatives to express $T$ as a derivative of $E$.

$$T = \frac{1}{\left( \frac{\partial S}{\partial E} \right)_V} = \frac{1}{1 / \left( \frac{\partial E}{\partial S} \right)_V} = \left( \frac{\partial E}{\partial S} \right)_V.$$

F. Solve the equation above for $P$ and use relations between partial derivatives to express it as a derivative of $E$.

$$P = T \left( \frac{\partial S}{\partial V} \right)_E = \frac{1}{\left( \frac{\partial S}{\partial E} \right)_V} \left( \frac{\partial S}{\partial V} \right)_E = - \left( \frac{\partial E}{\partial V} \right)_S.$$

The fundamental thermodynamic relation for a gas is

$$dE = TdS - PdV.$$

The heat capacity $C_X$ with some variable $X$ held fixed is defined to be the rate at which the system absorbs heat as the temperature changes:

$$C_X \equiv T \left( \frac{\partial S}{\partial T} \right)_X.$$

G. Express the heat capacity $C_V$ at constant volume in terms of a derivative of $E$.

thermodynamic relation $\Rightarrow$ $T = \left( \frac{\partial E}{\partial S} \right)_V$

$$C_V = T \left( \frac{\partial S}{\partial T} \right)_V = \left( \frac{\partial E}{\partial S} \right)_V \left( \frac{\partial S}{\partial T} \right)_V = \left( \frac{\partial E}{\partial T} \right)_V.$$

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