Relations between Partial Derivatives

Suppose the variables X, Y, and Z are related so that only two of them are independent. The differential of any one of them can be expressed in terms of differentials of the other two:

$$dX = \left(\frac{\partial X}{\partial Y}\right)_Z dY + \left(\frac{\partial X}{\partial Z}\right)_Y dZ,$$
$$dY = \left(\frac{\partial Y}{\partial X}\right)_Z dX + \left(\frac{\partial Y}{\partial Z}\right)_X dZ.$$

A. Insert the expression for dY into the equation for dX and collect the coefficients of dX and dZ on the right side.

$$dX = \left(\frac{\partial X}{\partial Y}\right)_Z \left[\left(\frac{\partial Y}{\partial X}\right)_Z dX + \left(\frac{\partial Y}{\partial Z}\right)_X dZ\right] + \left(\frac{\partial X}{\partial Z}\right)_Y dZ$$
$$= \left(\frac{\partial X}{\partial Y}\right)_Z \left(\frac{\partial Y}{\partial X}\right)_Z dX + \left[\left(\frac{\partial X}{\partial Y}\right)_Z \left(\frac{\partial Y}{\partial Z}\right)_X + \left(\frac{\partial X}{\partial Z}\right)_Y\right] dZ$$

The independence of the variables X and Z implies that the coefficients of dX and dZ on both sides of the equation must match.

B. Write down the resulting two equations involving partial derivatives.

$$1 = \left(\frac{\partial X}{\partial Y}\right)_Z \left(\frac{\partial Y}{\partial X}\right)_Z \qquad 0 = \left(\frac{\partial X}{\partial Y}\right)_Z \left(\frac{\partial Y}{\partial Z}\right)_X + \left(\frac{\partial X}{\partial Z}\right)_Y$$

C. Deduce the identity

$$\left(\frac{\partial Y}{\partial X}\right)_Z = \frac{1}{(\partial X/\partial Y)_Z}.$$

1st equation: $\left(\frac{\partial Y}{\partial X}\right)_Z = \frac{1}{\left(\frac{\partial X}{\partial Y}\right)_Z}$

D. Express $(\partial X/\partial Y)_Z$ in terms of a ratio of partial derivatives with respect to Z.

2nd equation: $\left(\frac{\partial X}{\partial Y}\right)_Z = -\frac{\left(\frac{\partial X}{\partial Z}\right)_Y}{\left(\frac{\partial Y}{\partial Z}\right)_X}$

$$= -\frac{1/(\partial Z/\partial X)_Y}{1/(\partial Z/\partial Y)_X} = -\frac{(\partial X/\partial Z)_Y}{(\partial Y/\partial Z)_X}$$

Compare your results with your neighbors.
Show your results to the instructor.

The energy E, volume V, and entropy S of a system are related by the equilibrium condition. In the microcanonical ensemble, the temperature T and the pressure P are naturally expressed in terms of derivatives of S:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_V, \qquad \frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_E.$$

E. Use relations between partial derivatives to express T as a derivative of E.

$$T = \frac{1}{(\partial S/\partial E)_V} = \frac{1}{1/(\partial E/\partial S)_V} = \left(\frac{\partial E}{\partial S}\right)_V$$

F. Solve the equation above for P and use relations between partial derivatives to express it as a derivative of E.

$$P = T\left(\frac{\partial S}{\partial V}\right)_E = \frac{1}{(\partial S/\partial E)_V}\left(\frac{\partial S}{\partial V}\right)_E = -\left(\frac{\partial E}{\partial V}\right)_S$$

The fundamental thermodynamic relation for a gas is

$$dE = TdS - PdV.$$

The heat capacity C_X with some variable X held fixed is defined to be the rate at which the system absorbs heat as the temperature changes:

$$C_X \equiv T\left(\frac{\partial S}{\partial T}\right)_X.$$

G. Express the heat capacity C_V at constant volume in terms of a derivative of E.

thermodynamic relation $\Longrightarrow$ $T = \left(\frac{\partial E}{\partial S}\right)_V$

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial E}{\partial S}\right)_V\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V$$

Compare your results with your neighbors.
Show your results to the instructor.