# Quantum Phase Space Volume

A particle with mass $m$ in one dimension is confined to the interval $0 < x < L$.
The Hamiltonian for a classical particle of momentum $p$ is $H(p) = p^2/2m$.

The Hamiltonian operator for a quantum particle is $\hat{H} = -(\hbar^2/2m)(\partial/\partial x)^2$.
The momentum operator is $\hat{P} = -i\hbar(\partial/\partial x)$.
The boundary conditions on the wavefunction are $\psi(x{=}0) = 0$, $\psi(x{=}L) = 0$.
A. Write down the independent eigenfunctions $\psi_n(x)$.

$$\psi_n(x) = \sin(n\pi x/L)$$

B. Specify all possible values of the quantum number $n$.

$$n = 1, 2, 3, \ldots$$

C. Identify the energy eigenvalue $\varepsilon_n$.

$$\mathcal{E}_n = -\frac{\hbar^2}{2m}\left(-\frac{n^2\pi^2}{L^2}\right) = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

D. Determine the action of the momentum operator $\hat{P}$ on $\psi_n(x)$.
Is it a momentum eigenstate?

$$\hat{P}\psi_n(x) = -i\hbar\frac{n\pi}{L}\cos(n\pi x/L) \Longrightarrow \text{not eigenstate of } \hat{P}$$

In the classical limit, the sum over quantum states can be reduced to an integral over phase space:

$$\sum_{n=1}^{\infty} f(\varepsilon_n) \longrightarrow \frac{1}{\omega_0}\int_{-\infty}^{+\infty} dp \int_0^L dx\, f(H(p)),$$

where $\omega_0$ is a normalizing factor with dimensions of angular momentum.
E. Simplify the sum on the left side in the classical limit by
(1) replacing the sum over $n$ by a discrete sum over a positive variable $p_n$ defined by $\varepsilon_n = p_n^2/2m$, $\Longrightarrow P_n = \sqrt{2m\mathcal{E}_n} = n\pi\hbar/L \Longrightarrow \Delta P_n = (\pi\hbar/L)\Delta n$
(2) approximating the sum over $p_n$ by a definite integral.

$$\sum_{n=1}^{\infty} f(\mathcal{E}_n) = \sum_{n=1}^{\infty}\Delta n\, f(\mathcal{E}_n) = \sum_{n=1}^{\infty}\left(\frac{L}{\pi\hbar}\Delta P_n\right) f\left(\frac{P_n^2}{2m}\right) = \frac{L}{\pi\hbar}\sum_{n=1}^{\infty}\Delta P_n\, f(P_n^2/2m)$$

$$\longrightarrow \frac{L}{\pi\hbar}\int_0^{\infty} dp\, f(p^2/2m) = \frac{L}{\pi\hbar}\cdot\frac{1}{2}\int_{-\infty}^{\infty} dp\, f\left(\frac{p^2}{2m}\right)$$

F. Deduce the quantum phase space factor $\omega_0$.

$$\frac{1}{\omega_0}\int_{-\infty}^{\infty} dp \int_0^L dx\, f(H) = \frac{1}{\omega_0} L \int_{-\infty}^{\infty} dp\, f(p^2/2m) \Longrightarrow \omega_0 = 2\pi\hbar$$

Compare your results with your neighbors.
Show your results to the instructor.

A particle with mass $m$ in one dimension is confined to a circle of circumference $L$ whose coordinate has the range $0 \le x < L$.

The Hamiltonian for a classical particle of momentum $p$ is $H(p) = p^2/2m$.

The Hamiltonian operator for a quantum particle is $\hat{H} = -(\hbar^2/2m)(\partial/\partial x)^2$.
The momentum operator is $\hat{P} = -i\hbar(\partial/\partial x)$.
The wavefunction satisfies periodic boundary conditions: $\psi(x{=}0) = \psi(x{=}L)$.
G. Write down the independent eigenfunctions $\psi_n(x)$.

$$\psi_n(x) = e^{2\pi i n x/L}$$

H. Specify all possible values of the quantum number $n$.

$$n = 0, \pm 1, \pm 2, \ldots$$

I. Identify the energy eigenvalue $\varepsilon_n$.

$$\varepsilon_n = -\frac{\hbar^2}{2m}\left(\frac{2\pi i n}{L}\right)^2 = \frac{2\pi^2\hbar^2 n^2}{mL^2}$$

J. Identify the momentum eigenvalue $p_n$.

$$p_n = -i\hbar\left(\frac{2\pi i n}{L}\right) = \frac{2\pi n\hbar}{L}$$

In the classical limit, the sum over quantum states can be reduced to an integral over phase space:

$$\sum_{n=\cancel{1}\,-\infty}^{\infty} f(\varepsilon_n) \longrightarrow \frac{1}{\omega_0}\int_{-\infty}^{+\infty} dp \int_0^L dx\, f(H(p)),$$

where $\omega_0$ is a normalizing factor with dimensions of angular momentum.
K. Simplify the sum on the left side in the classical limit by
(1) replacing the sum over $n$ by a discrete sum over the eigenvalue $p_n$, $\Delta p_n = \frac{2\pi\hbar}{L}\Delta n$
(2) approximating the sum over $p_n$ by a definite integral.

$$\sum_{n=-\infty}^{+\infty} f(\varepsilon_n) = \sum_{n=-\infty}^{\infty} \Delta n\, f(\varepsilon_n) = \sum_{n=-\infty}^{\infty}\left(\frac{L}{2\pi\hbar}\Delta p_n\right) f(p_n^2/2m) = \frac{L}{2\pi\hbar}\sum_{n=-\infty}^{\infty} \Delta p_n\, f(p_n^2/2m)$$

$$\longrightarrow \frac{L}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, f(p^2/2m)$$

L. Deduce the quantum phase space factor $\omega_0$.

$$\frac{1}{\omega_0}\int_{-\infty}^{\infty} dp \int_0^L dx\, f(H) = \frac{1}{\omega_0} L \int_{-\infty}^{\infty} dp\, f(p^2/2m) \Longrightarrow \omega_0 = 2\pi\hbar$$

Compare your results with your neighbors.
Show your results to the instructor.