

## Momentum and Coordinate Distributions

The partition function for a single atom at temperature  $T = 1/\beta$  confined to a volume  $V$  is

$$Z_1 = \frac{1}{(2\pi\hbar)^3} V \int d^3p e^{-\beta\epsilon(\vec{p})} = V \left( \frac{mkT}{2\pi\hbar^2} \right)^{3/2},$$

where  $\epsilon(\vec{p}) = \frac{1}{2m}\vec{p}^2$  is the energy of an atom with momentum  $\vec{p} = (p_x, p_y, p_z)$ .

The average energy  $\langle\epsilon\rangle$  of an atom can be obtained by differentiating the partition function:

$$\langle\epsilon\rangle = -\frac{\partial}{\partial\beta} \log Z_1.$$

A. Express  $\langle\epsilon\rangle$  as a ratio of momentum integrals.

$$\langle\epsilon\rangle = -\frac{\partial}{\partial\beta} \log \left( \int d^3p e^{-\beta\epsilon(\vec{p})} \right) = -\frac{1}{\int d^3p e^{-\beta\epsilon(\vec{p})}} \frac{\partial}{\partial\beta} \int d^3p e^{-\beta\epsilon(\vec{p})} = \frac{\int d^3p \epsilon(\vec{p}) e^{-\beta\epsilon(\vec{p})}}{\int d^3p e^{-\beta\epsilon(\vec{p})}}$$

B. Use the integration formulas below to calculate  $\langle\epsilon\rangle$ .

$$\langle\epsilon\rangle = \frac{4\pi \int_0^\infty p^2 dp \frac{p^2}{2m} e^{-\beta p^2/2m}}{4\pi \int_0^\infty p^2 dp e^{-\beta p^2/2m}} = \frac{\frac{1}{2m} \int_0^\infty dp p^4 e^{-\beta p^2/2m}}{\int_0^\infty dp p^2 e^{-\beta p^2/2m}} = \frac{\frac{1}{2m} 3\sqrt{\frac{\pi}{2}} \left(\frac{m}{\beta}\right)^{5/2}}{\sqrt{\frac{\pi}{2}} \left(\frac{m}{\beta}\right)^{3/2}} = \frac{3}{2m} \frac{m}{\beta}$$

The equipartition theorem implies that the average value of every quadratic term in the Hamiltonian is  $\frac{1}{2}kT$ .

C. Use the equipartition theorem to determine  $\langle\epsilon\rangle$ .

$$\langle\epsilon\rangle = 3 \cdot \frac{1}{2} kT = \frac{3}{2} kT$$

Suppose there are  $N$  atoms at temperature  $T$  confined to the volume  $V$ .

D. Deduce from  $Z_1$  the momentum distribution  $n(\vec{p})$  for the atoms.

(It is conventionally normalized so  $\int d^3p / (2\pi\hbar)^3 n(\vec{p}) = N/V$ .)

$$\int d^3p e^{-\beta\epsilon(\vec{p})} = 4\pi \int_0^\infty p^2 dp e^{-\beta p^2/2m} = 4\pi \sqrt{\frac{\pi}{2}} \left(\frac{m}{\beta}\right)^{3/2} = (2\pi m kT)^{3/2}$$

$$\int d^3p e^{-\beta\epsilon(\vec{p})} \times \frac{N/V}{(2\pi m kT)^{3/2}} = \frac{N}{V} \implies n(\vec{p}) = \frac{N/V}{(2\pi m kT)^{3/2}} e^{-\beta p^2/2m}$$

Some Gaussian integration formulas:

$$\int_0^\infty dp e^{-\beta p^2/2m} = \sqrt{\frac{\pi}{2}} \left(\frac{m}{\beta}\right)^{1/2}, \quad \int_0^\infty dp p^2 e^{-\beta p^2/2m} = \sqrt{\frac{\pi}{2}} \left(\frac{m}{\beta}\right)^{3/2}, \quad \int_0^\infty dp p^4 e^{-\beta p^2/2m} = 3\sqrt{\frac{\pi}{2}} \left(\frac{m}{\beta}\right)^{5/2}.$$

The partition function for a single atom at temperature  $T = 1/\beta$  trapped in a harmonic potential with angular frequency  $\omega$  is

$$Z_1 = \frac{1}{(2\pi\hbar)^3} \int d^3r \int d^3p e^{-\beta\epsilon(\vec{p}, \vec{r})} = \left(\frac{kT}{\hbar\omega}\right)^3,$$

where  $\epsilon(\vec{p}, \vec{r}) = \frac{1}{2m}\vec{p}^2 + \frac{1}{2}m\omega^2\vec{r}^2$  is the energy of an atom.

D. Calculate the average energy  $\langle\epsilon\rangle$  of an atom by differentiating the analytic result for the partition function.

$$\begin{aligned}\langle\epsilon\rangle &= -\frac{\partial}{\partial\beta} \log Z_1 = +T^2 \frac{\partial}{\partial T} \left(3 \log \frac{T}{\hbar\omega}\right) \\ &= 3T^2 \frac{1}{T} = 3T = 3kT\end{aligned}$$

E. Express the mean-square distance  $\langle r^2 \rangle$  of the atom from the center of the trapping potential as a ratio of integrals over the positions  $\vec{r}$ .

$$\langle r^2 \rangle = \frac{\int d^3r r^2 e^{-\beta \cdot \frac{1}{2} m \omega^2 r^2}}{\int d^3r e^{-\beta \cdot \frac{1}{2} m \omega^2 r^2}}$$

F. Use the equipartition theorem to deduce  $\langle r^2 \rangle$  as a function of  $T$ .

$$\left\langle \frac{1}{2} m \omega^2 \vec{r}^2 \right\rangle = 3 \cdot \frac{1}{2} kT \implies \langle r^2 \rangle = \frac{\frac{3}{2} kT}{\frac{1}{2} m \omega^2} = \frac{3kT}{m\omega^2}$$

Suppose there are  $N$  atoms at temperature  $T$  in the trapping potential.

G. Deduce from  $Z_1$  the local number density  $n(\vec{r})$  of the atoms.

(It is normalized so  $\int d^3r n(\vec{r}) = N$ .)

$$\frac{1}{(2\pi\hbar)^3} \int d^3p e^{-\beta p^2/2m} \int d^3r e^{-\beta \frac{1}{2} m \omega^2 r^2} = \left(\frac{kT}{\hbar\omega}\right)^3$$

$$\frac{(2\pi m kT)^{3/2}}{(2\pi\hbar)^3} \int d^3r e^{-\beta \cdot \frac{1}{2} m \omega^2 r^2} \times \frac{N}{(kT/\hbar\omega)^3} = N \implies n(\vec{r}) = N \left(\frac{m\omega^2}{kT}\right)^{3/2} e^{-\beta \cdot \frac{1}{2} m \omega^2 r^2}$$

The mean-square distance  $\langle r^2 \rangle$  can be used as a thermometer for the many-body system. It can be determined from measurements of  $n(\vec{r})$ .

H. Express the temperature  $T$  in terms of an integral involving the number density  $n(\vec{r})$ .

$$kT = \frac{m\omega^2}{3} \langle r^2 \rangle = \frac{m\omega^2}{3} \frac{\int d^3r r^2 n(r)}{N}$$