Density Matrices for Spin States

The basis for the spin-angular-momentum quantum states of a single particle with spin $\frac{1}{2}$ consists of two orthonormal states: $|\uparrow\rangle$ and $|\downarrow\rangle$.

If two such particles are distinguishable, the direct-product basis for their spin states consists of 4 orthonormal states: $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$.

If the two particles are identical bosons, the orthonormal basis consists of 3 states: $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|\downarrow\downarrow\rangle$.

If the two particles are identical fermions, the orthonormal basis consists of a single state: $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

A. Write down the orthonormal basis of 8 spin states for 3 distinguishable particles.

$|\uparrow\uparrow\uparrow\rangle \quad |\uparrow\uparrow\downarrow\rangle \quad |\uparrow\downarrow\uparrow\rangle \quad |\downarrow\uparrow\uparrow\rangle \quad |\uparrow\downarrow\downarrow\rangle \quad |\downarrow\uparrow\downarrow\rangle \quad |\downarrow\downarrow\uparrow\rangle \quad |\downarrow\downarrow\downarrow\rangle$

B. Write down the orthonormal basis of 4 spin states for 3 identical bosons.

$|\uparrow\uparrow\uparrow\rangle \quad \frac{1}{\sqrt{3}}\left(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\right) \quad \frac{1}{\sqrt{3}}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right) \quad |\downarrow\downarrow\downarrow\rangle$

C. Write down the orthonormal basis of spin states for 3 identical fermions (if there are any).

there are none

The energies of the single-particle spin states are

$$|\uparrow\rangle : \varepsilon, \qquad |\downarrow\rangle : 0.$$

D. Assuming the particles are noninteracting, give the energy of each of the 4 basis states for 2 distinguishable particles.

$|\uparrow\uparrow\rangle : 2\varepsilon \quad |\uparrow\downarrow\rangle : \varepsilon \quad |\downarrow\uparrow\rangle : \varepsilon \quad |\downarrow\downarrow\rangle : 0$

E. Give the energy of each of the 3 basis states for 2 identical bosons.

$|\uparrow\uparrow\rangle : 2\varepsilon \quad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) : \varepsilon \quad |\downarrow\downarrow\rangle : 0$

F. Give the energy of the single basis state for 2 identical fermions.

$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) : \varepsilon$

The density matrix for the microcanonical ensemble with total energy ε for the spin states of 2 distinguishable particles is

$$\hat{\rho} = \frac{1}{2}\Big(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|\Big).$$

G. Verify that $\mathrm{Tr}[\hat{\rho}] = 1$.

$$\mathrm{Tr}[\hat{\rho}] = \frac{1}{2}\Big(\mathrm{Tr}[|\uparrow\downarrow\rangle\langle\uparrow\downarrow|] + \mathrm{Tr}[|\downarrow\uparrow\rangle\langle\downarrow\uparrow|]\Big)$$
$$= \frac{1}{2}\Big(\mathrm{Tr}[\langle\uparrow\downarrow|\uparrow\downarrow\rangle] + \mathrm{Tr}[\langle\downarrow\uparrow|\downarrow\uparrow\rangle]\Big) = \frac{1}{2}(1+1) = 1$$

H. What is the density matrix for the microcanonical ensemble with total energy ε for the spin states of 2 identical bosons?

$$\hat{\rho} = \frac{1}{2}\big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big)\big(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|\big)$$

I. What is the density matrix for the microcanonical ensemble with total energy ε for the spin states of 2 identical fermions?

$$\hat{\rho} = \frac{1}{2}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big)\big(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\big)$$

The density matrix for the canonical ensemble with temperature $T = 1/\beta$ for the spin states of 2 distinguishable particles is

$$\hat{\rho} = \frac{1}{(1+e^{-\beta\varepsilon})^2}\Big(e^{-2\beta\varepsilon}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + e^{-\beta\varepsilon}|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + e^{-\beta\varepsilon}|\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\Big).$$

J. Verify that $\mathrm{Tr}[\hat{\rho}] = 1$.

$$\mathrm{Tr}[\hat{\rho}] = \frac{1}{(1+e^{-\beta\varepsilon})^2}\Big(e^{-2\beta\varepsilon}\mathrm{Tr}[|\uparrow\uparrow\rangle\langle\uparrow\uparrow|] + e^{-\beta\varepsilon}\mathrm{Tr}[|\uparrow\downarrow\rangle\langle\uparrow\downarrow|]$$
$$+ e^{-\beta\varepsilon}\mathrm{Tr}[|\downarrow\uparrow\rangle\langle\downarrow\uparrow|] + \mathrm{Tr}[|\downarrow\downarrow\rangle\langle\downarrow\downarrow|]\Big)$$
$$= \frac{1}{(1+e^{-\beta\varepsilon})^2}\big(e^{-2\beta\varepsilon} + e^{-\beta\varepsilon} + e^{-\beta\varepsilon} + 1\big) = 1$$

K. What is the density matrix for the canonical ensemble for the spin states of 2 identical bosons?

$$\hat{\rho} = \frac{1}{1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}}\Big(e^{-2\beta\varepsilon}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + e^{-\beta\varepsilon}\tfrac{1}{2}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle) + |\downarrow\downarrow\rangle\Big)$$

L. What is the density matrix for the canonical ensemble for the spin states of 2 identical fermions?

$$\hat{\rho} = \frac{1}{2}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big)\big(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\big)$$