Occupation Numbers for Spin States

The basis for the spin-angular-momentum quantum states of a single particle with spin $\frac{1}{2}$ consists of two orthonormal states: $|\uparrow\rangle$ and $|\downarrow\rangle$.
If there is more than one such particle, the basis of spin states can be expressed in terms of direct products of $|\uparrow\rangle$ and $|\downarrow\rangle$.

If the particles are identical bosons or identical fermions, a more convenient orthonormal basis consists of states $|n_\uparrow, n_\downarrow\rangle$ labeled by occupation numbers.
If the particles are identical bosons, $n_\uparrow$ and $n_\downarrow$ can be $0, 1, 2, \ldots$.
If the particles are identical fermions, $n_\uparrow$ and $n_\downarrow$ can be $0, 1$.

If there are N identical bosons or N identical fermions, the occupation numbers add up to N: $n_\uparrow + n_\downarrow = N$.

The state $|0,0\rangle$ can be interpreted as the vacuum state $|\varnothing\rangle$.
The spin states for a single particle are $|1,0\rangle = |\uparrow\rangle$ and $|0,1\rangle = |\downarrow\rangle$.

If the two particles are identical bosons, the orthonormal basis consists of 3 states: $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|\downarrow\downarrow\rangle$.
A. Express each of these basis states as an occupation-number state $|n_\uparrow, n_\downarrow\rangle$.

$|2,0\rangle = |\uparrow\uparrow\rangle \quad |1,1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \quad |0,2\rangle = |\downarrow\downarrow\rangle$

If the two particles are identical fermions, the orthonormal basis consists of a single state: $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.
B. Express this state as an occupation-number state $|n_\uparrow, n_\downarrow\rangle$.

$|1,1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

C. Write down the orthonormal basis of 4 spin states for 3 identical bosons in terms of direct products of $|\uparrow\rangle$ and $|\downarrow\rangle$.

$|\uparrow\uparrow\uparrow\rangle \quad \frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) \quad \frac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \quad |\downarrow\downarrow\downarrow\rangle$

D. Express each of these basis states as an occupation-number state $|n_\uparrow, n_\downarrow\rangle$.

$|3,0\rangle \quad |2,1\rangle \quad |1,2\rangle \quad |0,3\rangle$

E. Write down the orthonormal basis of 5 spin states for 4 identical bosons in terms of occupation-number states $|n_\uparrow, n_\downarrow\rangle$.

$|4,0\rangle \quad |3,1\rangle \quad |2,2\rangle \quad |1,3\rangle \quad |0,4\rangle$

The energies of the single-particle quantum states are

$$|\uparrow\rangle : \ \varepsilon, \qquad |\downarrow\rangle : \ 0.$$

F. Assuming the particles are noninteracting, what is the energy of the occupation-number state $|n_\uparrow, n_\downarrow\rangle$?

$$n_\uparrow \cdot \varepsilon + n_\downarrow \cdot 0 = n_\uparrow \varepsilon$$

The density matrix for 2 identical bosons in the canonical ensemble with temperature $T = 1/\beta$ is

$$\hat{\rho} = \frac{1}{1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}}\Big(|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + e^{-\beta\varepsilon}\frac{1}{2}\big(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\big)\big(\langle\uparrow\downarrow|+\langle\downarrow\uparrow|\big) + e^{-2\beta\varepsilon}|\uparrow\uparrow\rangle\langle\uparrow\uparrow|\Big).$$

G. Express this density matrix in terms of occupation-number states.

$$\hat{\rho} = \frac{1}{1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}}\Big(|0,2\rangle\langle 0,2| + e^{-\beta\varepsilon}|1,1\rangle\langle 1,1| + e^{-2\beta\varepsilon}|2,0\rangle\langle 2,0|\Big)$$

H. Write down the density matrix for 3 identical bosons in the canonical ensemble in terms of occupation-number states.

$$\hat{\rho} = \frac{1}{1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}+e^{-3\beta\varepsilon}}\Big(|0,3\rangle\langle 3,0| + e^{-\beta\varepsilon}|1,2\rangle\langle 1,2| + e^{-2\beta\varepsilon}|2,1\rangle\langle 2,1| + e^{-3\beta\varepsilon}|3,0\rangle\langle 3,0|\Big)$$

The density matrix for identical fermions in the grand canonical ensemble with temperature $T = 1/\beta$ and chemical potential μ is

$$\hat{\rho} = \frac{1}{(1+e^{\beta\mu})(1+e^{\beta\mu-\beta\varepsilon})} \times\Big(|\emptyset\rangle\langle\emptyset| + e^{\beta\mu-\beta\varepsilon}|\uparrow\rangle\langle\uparrow| + e^{\beta\mu}|\downarrow\rangle\langle\downarrow| + e^{2\beta\mu-\beta\varepsilon}\frac{1}{2}\big(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\big)\big(\langle\uparrow\downarrow|-\langle\downarrow\uparrow|\big)\Big).$$

I. Verify that $\mathrm{Tr}[\hat{\rho}] = 1$.

$$\mathrm{Tr}[\hat{\rho}] = \frac{1}{(1+e^{\beta\mu})(1+e^{\beta\mu-\beta\varepsilon})}\Big[1+e^{\beta\mu-\beta\varepsilon}+e^{\beta\mu}+e^{2\beta\mu-\beta\varepsilon}\Big] = 1$$

J. Express this density matrix in terms of occupation-number states.

$$\hat{\rho} = \frac{1}{(1+e^{\beta\mu})(1+e^{\beta\mu-\beta\varepsilon})}\Big[|0,0\rangle\langle 0,0| + e^{\beta\mu-\beta\varepsilon}|1,0\rangle\langle 1,0| + e^{\beta\mu}|0,1\rangle\langle 0,1| + e^{2\beta\mu-\beta\varepsilon}|1,1\rangle\langle 1,1|\Big]$$