Bosonic Trapped Atoms in 2D

The energy of an atom in a 2-dimensional harmonic trap is

$$\varepsilon(\vec{p},\vec{r}) = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2+y^2\right).$$

Suppose the atoms in the trap are in thermal equilibrium at temperature $T = 1/\beta$ and in chemical equilibrium at chemical potential μ.
If the atoms are identical bosons and T is greater than the critical temperature T_c for Bose-Einstein condensation, the number N of atoms in the trap is

$$N = \frac{1}{(2\pi\hbar)^2}\int d^2r \int d^2p \, \frac{1}{\exp(\beta[\varepsilon(\vec{p},\vec{r})-\mu])-1}.$$

It is convenient to change variables from (x, y, p_x, p_y) to $(P_1 = p_x, P_2 = p_y, P_3 = m\omega x, P_4 = m\omega y)$.
A. Express the energy $\varepsilon(\vec{p},\vec{r})$ as a function of P_1, P_2, P_3, and P_4.

$$\mathcal{E} = \frac{1}{2m}\left(P_1^2 + P_2^2 + P_3^2 + P_4^2\right)$$

B. Express N in terms of an integral over the 4-dimensional vector $\vec{P}$.

$$dx = \frac{1}{m\omega}dP_3 \quad dy = \frac{1}{m\omega}dP_4 \qquad d^2r\,d^2p = \frac{1}{(m\omega)^2}d^4P$$

$$N = \frac{1}{(2\pi\hbar)^2}\frac{1}{(m\omega)^2}\int d^4P \, \frac{1}{e^{\beta(\vec{P}^2/2m-\mu)}-1}$$

The differential volume in D dimensions is $d^Dx = d\Omega_D\, r^{D-1}dr$. The integral over angles is

$$\Omega_D = \int d\Omega_D = 2\pi^{D/2}/\Gamma(D/2).$$

C. Use this to calculate the volume of a ball of radius R in D dimensions.

$$\int d^Dr\ \Theta(R^2-r^2) = \int d\Omega_D \int_0^R r^{D-1}dr = 2\frac{\pi^{D/2}}{\Gamma(D/2)}\frac{R^D}{D} = \frac{\pi^{D/2}}{\Gamma(\frac{D}{2}+1)}R^D$$

The gamma function satisfies $\Gamma(z+1) = z\,\Gamma(z)$, $\Gamma(1) = 1$, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
D. Express N in terms of an integral over the length $P = |\vec{P}|$ of the 4-dimensional vector.

$$N = \frac{1}{(2\pi\hbar m\omega)^2}\underbrace{\int d\Omega_4}_{\frac{2\pi^2}{\Gamma(2)} = 2\pi^2}\int_0^\infty P^3 dP\, \frac{1}{e^{\beta(P^2/2m-\mu)}-1}$$

Above the critical temperature for Bose-Einstein condensation, the number N of atoms can be expressed in terms of an integral over a dimensionless variable:

$$N = \left(\frac{T}{\hbar\omega}\right)^2 \int_0^\infty dx \frac{x}{\exp(x - \beta\mu) - 1} = \left(\frac{T}{\hbar\omega}\right)^2 g_2(e^{\beta\mu}) \qquad T \geq T_c.$$

Bose-Einstein condensation begins at the temperature T_c where the chemical potential μ increases to 0.

E. Write down the above equation for N at the critical temperature T_c.

$$T = T_c \Longrightarrow \mu = 0$$

$$N = \left(\frac{T_c}{\hbar\omega}\right)^2 g_2(1)$$

F. Using $g_2(1) = \zeta_2$, deduce the critical temperature T_c as a function of N.

$$N = \left(\frac{T_c}{\hbar\omega}\right)^2 \zeta_2 \Longrightarrow T_c = \left(\frac{1}{\zeta_2} N\right)^{1/2} \hbar\omega$$

Below the critical temperature, the number N of atoms can be expressed as

$$N = N_0(T) + \left(\frac{T}{\hbar\omega}\right)^2 \int_0^\infty dx \frac{x}{\exp(x) - 1} = N_0(T) + \left(\frac{T}{\hbar\omega}\right)^2 \zeta_2 \qquad T \leq T_c,$$

where $N_0(T)$ is the number of atoms in the condensate.

G. Use the equation for N at T_c in part E to express the last term in terms of N and the temperature ratio T/T_c.

$$\left(\frac{T}{\hbar\omega}\right)^2 \zeta_2 = \left(\frac{T}{\hbar\omega}\right)^2 \zeta_2 \times \frac{N/\zeta_2}{(T_c/\hbar\omega)^2} = N\left(\frac{T}{T_c}\right)^2$$

H. Make this substitution for the last term in the equation for N, and then solve it for $N_0(T)$, expressing it as the product of N and a function of T.

$$N = N_0(T) + N\left(\frac{T}{T_c}\right)^2 \Longrightarrow N_0(T) = N\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

I. Deduce the number of thermal atoms in the condensate.

$$N_{thermal} = \left(\frac{T}{T_c}\right)^2 N$$