Bosonic Trapped Atoms

A large number of identical bosonic atoms are trapped in a 3-dimensional harmonic oscillaor potential. The atoms are in thermal equilibrium at temperature $T = 1/\beta$ and in chemical equilibrium at chemical potential μ.
If T is greater than the critical temperature T_c for Bose-Einstein condensation, the number N of atoms can be expressed as

$$N = \frac{1}{2}\left(\frac{T}{\hbar\omega}\right)^3 \int_0^\infty dx \frac{x^2}{\exp(x - \beta\mu) - 1} = \left(\frac{T}{\hbar\omega}\right)^3 g_3\left(e^{\beta\mu}\right).$$

Bose-Einstein condensation begins when T decreases to the temperature T_c where the chemical potential μ increases to 0.
A. Using $g_3(1) = \zeta_3$, deduce the critical temperature T_c.

$$N = \left(\frac{T_c}{\hbar\omega}\right)^3 g_3(1) = \zeta_3\left(\frac{T_c}{\hbar\omega}\right)^3 \qquad T_c = \left(\frac{1}{\zeta_3}N\right)^{1/3}\hbar\omega$$

B. For $T < T_c$, express the number N of atoms in terms of the temperature T and the number N_0 of atoms in the condensate.

$$N = N_0 + \zeta_3\left(\frac{T}{\hbar\omega}\right)^3$$

C. Deduce the critical temperature T_c by taking the limit of this equation as T approaches T_c from below.

$$N = 0 + \zeta_3\left(\frac{T_c}{\hbar\omega}\right)^3 \Longrightarrow T_c = \left(\frac{1}{\zeta_3}N\right)^{1/3}\hbar\omega$$

$T < T_c$
D. Express the equation for N for $\cancel{T > T_c}$ in terms of N_0 and T/T_c.

$$N = N_0 + \zeta_3\left(\frac{T}{\hbar\omega}\right)^3 \times \frac{N/\zeta_3}{(T_c/\hbar\omega)^3} = N_0 + N\left(\frac{T}{T_c}\right)^3$$

E. Solve for the number N_0 of condensate atoms as a function of N and T/T_c.

$$N_0 = N\left[1 - \left(\frac{T}{T_c}\right)^3\right]$$

F. Deduce the number N_{th} of thermal atoms as a function of N and T/T_c.

$$N_{th} = N\left(\frac{T}{T_c}\right)^3$$

At temperatures $T > T_c$, the number N of atoms can be expressed as an integral over the coordinates and the momentum components of a single atom. N can also be expressed as an integral of the local number density: $N = \int d^3r\, n(\vec{r})$.

For $T \gg T_c$, the expression for the number of atoms is

$$N = \frac{e^{\beta\mu}}{(2\pi\hbar)^3} \int d^3r \int d^3p\, e^{-\beta(p^2/2m + m\omega^2 r^2/2)} = \frac{e^{\beta\mu}}{\lambda_T^3} \int d^3r\, e^{-\beta m\omega^2 r^2/2},$$

where $\beta = 1/T$ and $\lambda_T = (2\pi\hbar^2/mT)^{1/2}$.

G. Deduce $n(r)$ as a function of the distance r from the center of the trap.

$$n(r) = \frac{e^{\beta\mu}}{\lambda_T^3} e^{-\beta m\omega^2 r^2/2}$$

H. Estimate the average radius $\langle r\rangle$ of the thermal cloud as a function of T and/or N.

$$\langle r\rangle \sim \frac{1}{\sqrt{\beta m\omega^2}}$$

At $T = T_c = 1/\beta_c$, the expression for the number of atoms is

$$N = \frac{1}{(2\pi\hbar)^3} \int d^3r \int d^3p \frac{1}{e^{\beta(p^2/2m + m\omega^2 r^2/2} - 1} = \frac{1}{\lambda_{T_c}^3} \int d^3r\, g_{3/2}\left(e^{-\beta_c m\omega^2 r^2/2}\right).$$

I. Deduce $n(r)$ as a function of r.

$$n(r) = \frac{1}{\lambda_T^3} g_{3/2}\left(e^{-\beta_c m\omega^2 r^2/2}\right)$$

J. Estimate $\langle r\rangle$ of the thermal cloud as a function of T and/or N. (The function $n(r)$ has only one length scale.)

$$\langle r\rangle \sim \frac{1}{\sqrt{\beta_c m\omega^2}} = \sqrt{\frac{\left(\frac{1}{\zeta_3} N\right)^{1/3} \hbar\omega}{m\omega^2}} \sim N^{1/6} \sqrt{\frac{\hbar}{m\omega}}$$

At $T = 0$, all N atoms are in the ground state of the harmonic oscillator, whose normalized wavefunction is

$$\psi_0(\vec{r}) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-(m\omega/\hbar)r^2/2}.$$

I. Deduce $n(r)$ as a function of r.

$$n(r) = N\,|\psi_0(r)|^2 = N\left(\frac{m\omega}{\pi\hbar}\right)^{3/2} e^{-(m\omega/\hbar)r^2}$$

H. Estimate the average radius $\langle r\rangle$ of the BEC as a function of T and/or N.

$$\langle r\rangle \sim \frac{1}{\sqrt{m\omega/\hbar}} = \sqrt{\frac{\hbar}{m\omega}}$$